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INTEGRATED GRADIENTS: A DERIVATION OF SOME DIFFERENCE FORMS FOR THE EQUATION OF MOTION FOR COMPRESSIBLE FLOW IN TWO-DIMENSIONAL LAGRANGIAN HYDRODYNAMICS, USING INTEGRATION OF PRESSURES OVER SURFACES

by

Philip L. Browne

ABSTRACT

This paper describes a method of deriving gradients (that is, accelerations) for difference calculations of the equations of motion (momentum conservation) in two-dimensional Lagrangian meshes in an r-z coordinate system. The method basically considers various ways of defining the masses associated with each vertex and methods of integrating pressures over the surfaces of those masses, and then combining them in various ways to conserve momentum transfer between vertices. These gradients are derived analytically for planes, cylinders, and spheres to test for uniform motion. All results described have been tested with actual numerical calculations.

I. INTRODUCTION

Most difference equations are derived by approximating differential equations. However, the differential equations themselves are derived by taking a small discrete element, applying the physics, and then allowing the proper quantities to shrink to zero.

By integrated gradients we mean taking a small element and applying the physics to it to get the difference equation directly.* This seems like a good approach because we leave out the step of shrinking quantities to zero (which often drops out terms that are important as the difference equations are applied over and over again in time-dependent problems). This is the fundamental approach used in this report.

This work could result in a number of "models" of a fluid in a two-dimensional Lagrangian mesh and the corresponding difference equations derived therefrom. We have worked mostly with three models, which are named:
(1) according to the way the pressure integral over a surface is obtained, denoted by

IGT = integrated gradient total,

IGA = integrated gradient average, and

FGI = force gradient I (a name given by S. R. Orr), and

(2) according to the way that the mass associated with a vertex is chosen, denoted by q = one quarter of the zone mass and MAC-0 = the mass of subzones formed by joining the midpoints of the sides at the start of the problem. Averaging the four corners gives the same result.

From a combination of the conceptual and analytic arguments in this report plus investigations of rezoning,**[1-3] viscosity,†[3] and performance in a number of relatively simple problems, we prefer the MAC-0 method for choosing the masses and we lean toward the IGA method for taking pressure integrals. This model gives complete, exact conservation of the mass, momentum, and energy of the model as they are transferred between adjacent points on zones of the mesh, with no overlapping of the masses associated with adjacent points of the mesh. Thus, one knows exactly what mass is associated with each point in the mesh. This model made possible the derivation of a rezoning scheme[1-3] with conservation of mass, momentum, and energy. It also made the derivation

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*This concept gradually evolved during discussions between the author and G. N. White in October 1961.

**Conducted by the author and Karl B. Wallick, Group X-6, in 1964-1965.

†Studies by the author and Karl B. Wallick, Group X-6, in 1964-1965 and by the author and Patrick J. Blewett, X-5, in 1966-1967.

of the accelerations for boundary points and other special situations relatively straightforward. Finally, numerical calculations with this method indicate that it tends to delay distortions and oscillations, especially at free surfaces.

The IGT method will not give spherical motion in a spherical problem with equal angular spacing. It is interesting to note that a mass weighted average of the force terms (F/M) in IGA gives IGT.

The FGI-q method has been used for many years by S. R. Orr and coworkers with good success, but the rezone in that code had to give up conservation of some of the quantities.

The IGA-MACO method of doing an automatic rezone[1-2] has been successfully used by Karl Wallick and the current author in a large code which uses the Schulz[4] gradients for the hydrodynamics.

II. THE INTEGRAL METHOD

In a continuous medium the differential equation of motion for a compressible flow without viscosity is given by

$$\rho\vec{a} = -\nabla P \quad , \tag{1}$$

where

ρ = density

P = pressure

$\vec{a}$ = acceleration = $\frac{d\vec{v}}{dt}$ = time derivative of the velocity, $\vec{v}$, of a particle as one moves along with the particle.

If we integrate equation (1) over any volume, V, enclosed by a surface, S, we get

$$\int_V \rho\vec{a}dV = -\int_V (\nabla P)dV = -\int_S Pd\vec{S} \quad , \tag{2}$$

where $d\vec{S}$ is a vector representing a surface element and has the direction of the outward drawn normal.[5] A common method of deriving difference equations for numerical work is to start with the differential equation, (1), or variations of it, and to approximate the derivatives by differences. As an alternative, one might begin with an integral form, such as (2), and try to work directly toward the difference equations. This we have done. The methods for deriving integrated gradients proposed in this report are general and can be applied to many types of Lagrangian meshes. However, for a number of reasons, this work deals exclusively with quadrilateral meshes, that is, meshes in which each zone has four sides and four corners or vertices. Also, except for special boundary cases, each vertex is a corner for each of the four zones surrounding it. This type of zoning seems to have many advantages: it is easier to fit the requirements of the zoning to the logic of a computer; it is generally easier to adapt such a mesh to the types of configurations one wishes to work with; and, finally, we have had more experience with this type of mesh.[6,7] We have done some experimentation with triangular meshes and found them generally unsatisfactory from both a computational and physical standpoint.

III. BASIC DEFINITIONS

Let us assume that we have a Lagrangian mesh[6,7] imbedded in a fluid (Fig. 1), and that we are looking at a point O(r,z) where L zones have a common vertex.

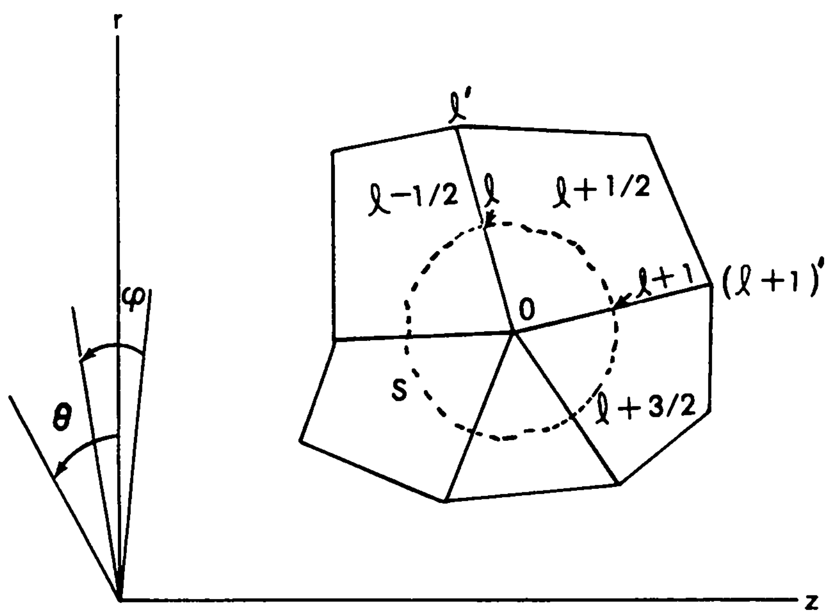

Fig. 1. A typical vertex with the adjacent zones and vertices.

Recall that this picture represents a figure of revolution about the z axis, which means that each zone represents an element of volume and that all scalar functions, such as ρ, P, etc., are independent of Θ. The vectors such as $\vec{a}$ and $d\vec{S}$ vary direction with Θ, but their magnitudes remain constant. We cannot take a revolution of 2π about the z axis, for then the r component of the surface integral $\int PdS$ would vanish. Hence, we consider a revolution of small angle, ϕ, about the z axis, which produces a kind of wedge-shaped volume as viewed from along the z axis. The forces on the sides of this wedge must be considered.

We will identify quantities along the boundary between zones by a subscript ℓ and quantities in the zone by subscript $\ell+1/2$. The next vertex out along side ℓ will be identified by ℓ'. We assume that in each zone there is a uniform pressure $P_{\ell+1/2}$ and density $\rho_{\ell+1/2}$. These quantities may therefore be discontinuous across the boundaries between zones.

Now, to find an acceleration at 0, let us draw any closed surface, S, about 0 (dashed lines in Fig. 1) and consider how to apply the integral method of (2) to the material enclosed by S. The most logical method would seem to be one in which S would be chosen in such a way that 0 would be at the center of mass of the material enclosed by S.[8] This method appears very interesting, but formidable. We have attempted to devise such a scheme in one-dimensional problems, with inconclusive results mainly because of its complexity.

As will be shown later (Theorem 8), when the pressure in a zone is considered to be uniform, the <u>total surface integral, $\int P d\vec{S}$, over that zone is independent of the path chosen between ℓ, $\ell+1$</u>. This means that once having selected the ℓ's, for different S having center of mass at 0,

$$\int_V \rho \vec{a} dV = C \quad .$$

Using a mean value type argument, if we assume that there exists an $\vec{a}_0$ such that $\int_V \rho \vec{a} dV = \vec{a}_0 \int_V \rho dV = \vec{a}_0 M$ we can then say that

$$\vec{a}_0 = \frac{C}{M} \quad .$$

In other words, since M changes with S, while C does not, it is apparent that a_0 is not unique even though an S is chosen to give the center of mass at O. This implies that other additional criteria might be needed to select an S which gives a useful value for $\vec{a}_0$.

The simpler methods we have selected (for dealing primarily with a quadrilateral mesh) we shall call:

IGT - Integrated Gradient, Total

IGA - Integrated Gradient, Average

FG - Force Gradient.

The use of the word gradient in these names is a misnomer for we are deriving accelerations rather than gradients. However, we continue to use the names above because a great deal of analytic work and computation has been done using this nomenclature. These gradients are herewith described.

There are several common procedures which we must carry out in all the methods. For each zone or group of zones about a vertex, $\vec{a}$ in equation (2) is taken out of the integral so that

$$\vec{a}_{\ell+1/2} = \left(-\int_S P d\vec{S}\right) / \left(\int_V \rho dV\right)_{\ell+1/2} = \vec{F}_{\ell+1/2}/M_{\ell+1/2} \quad . \tag{3}$$

As mentioned before, this is equivalent to making a mean value type argument. Points ℓ (the intersections of S with the common boundaries between zones) and ℓ' must be defined. As mentioned earlier, the choice of S inside the zone will be shown to have no effect on $\vec{F}_{\ell+(1/2)} = -\int_S P d\vec{S}$, but it will obviously affect the value of $M_{\ell+1/2} = \int_V \rho dV$. (4)

The pressures, $P_\ell^{\ell+1/2}$ and $P_\ell^{\ell-1/2}$, along side ℓ used in calculating

$\vec{F} = \int P d\vec{S}$ for zones $\ell+1/2$ and $\ell-1/2$ need to be defined. (5)

S must be defined between point ℓ, $(\ell+1)$ in order to define $M_{\ell+1/2}$. (6)

We now briefly describe the three general types of gradients.

(IGT) Integrated Gradient, Total. This method derives an acceleration for point 0 from a total force on all zones around a point and the total mass involved, that is,

$$\vec{a} = \sum_{\ell=1}^{L} \vec{F}_{\ell+1/2} / \sum_{\ell=1}^{L} M_{\ell+1/2} \quad . \tag{7}$$

(IGA) Integrated Gradient, Average. This method evaluates an acceleration, $\vec{a}_{\ell+1/2}$, from each zone around a point and then averages these to get the point acceleration, that is,

$$\vec{a} = \frac{1}{L} \sum_{\ell=1}^{L} \vec{a}_{\ell+1/2} = \frac{1}{L} \sum_{\ell=1}^{L} (\vec{F}_{\ell+1/2}/M_{\ell+1/2}) \quad . \tag{8}$$

It is interesting that since $\vec{a}_{\ell+1/2}$ is really the acceleration of the center of mass of $M_{\ell+1/2}$, one could argue for a mass weighted average of the individual $\vec{a}_{\ell+1/2}$, that is,

$$\vec{a} = \sum_{\ell=1}^{L} (M_{\ell+1/2}\, \vec{a}_{\ell+1/2}) / \sum_{\ell=1}^{L} (M_{\ell+1/2}) = \sum_{\ell=1}^{L} (\vec{F}_{\ell+1/2}) / \sum_{\ell=1}^{L} (M_{\ell+1/2}) \ , \tag{8a}$$

which gives the acceleration at the center of mass of the material enclosed by S. This is none other than IGT. Since a mass weighted average seems preferable to an arithmetic average, we tend to lean toward IGT as being more intuitively appealing. But IGT has its numerical disadvantages.

(FG) Force Gradient. This method obtains an acceleration for each pair of zones and then averages these accelerations, that is,

$$\vec{a} = \frac{1}{L} \sum_{\ell=1}^{L} \vec{a}_{\ell} = \frac{1}{L} \sum_{\ell=1}^{L} \frac{\vec{F}_{\ell-1/2} + \vec{F}_{\ell+1/2}}{M_{\ell-1/2} + M_{\ell+1/2}} \quad . \tag{9}$$

Under certain assumptions about ℓ', one can obtain a gradient called FGI which was derived by other (nonintegral) methods and was used by S. R. Orr in a large code.

There are many other, more complicated ways in which one could combine forces and masses to get other acceleration formulae, but we have confined our efforts to the three methods described above.

IV. CONVENTIONS AND BASIC THEOREMS

When using cylindrical coordinates (r, Θ, z) to deal with a system which has cylindrical symmetry, note that although scaler functions and magnitudes of vectors are independent of Θ, the directions of the vectors may vary with Θ. If one wishes to perform integrations involving vector quantities (such as $\int P d\vec{S}$), these variations in direction should be taken into account. One way to do this is to write the vectors in terms of a set of unit vectors $(\vec{r}_1, \vec{\Theta}_1, \vec{z}_1)$ in the cylindrical system (Fig. 2).

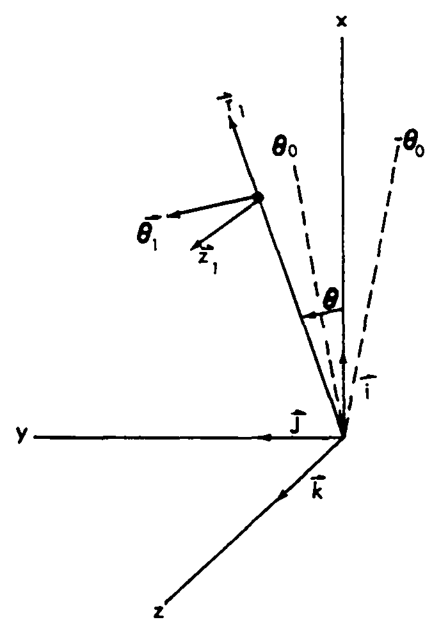

Fig. 2. Unit vectors in the cylindrical and Cartesian systems.

The moving unit vectors $(\vec{r}_1, \vec{\Theta}_1, \vec{z}_1)$ can then be expressed in terms of the fixed unit vectors $(\vec{i}, \vec{j}, \vec{k})$ in a Cartesian system by

$$\vec{r}_1 = \vec{i} \cos \Theta + \vec{j} \sin \Theta$$

$$\vec{\Theta}_1 = \vec{i} \, (- \sin \Theta) + \vec{j} \cos \Theta$$

$$\vec{z}_1 = \vec{k} \quad . \tag{10}$$

Since $(\vec{i}, \vec{j}, \vec{k})$ have constant magnitude and direction in space, they can be removed from the integrals.

As mentioned earlier, the volumes which we consider will be wedge-shaped slices produced by rotating the mesh through a small angle, ϕ, about the z-axis. If ϕ is considered to extend from $-\Theta_0$ to Θ_0 (Fig. 2), it will be useful to recall the relations

$$\int_{-\Theta_0}^{\Theta_0} \cos \Theta \, d\Theta = 2 \sin \Theta_0 \approx 2\Theta_0 = \phi$$

$$\int_{-\Theta_0}^{\Theta_0} \sin \Theta \, d\Theta = 0$$

$$\int_{-\Theta_0}^{\Theta_0} d\Theta = 2\Theta_0 = \phi \quad . \tag{11}$$

Theorem 1. If a function, $P(r, z)$, and any curve, $(\ell, \ell+1)$, are defined in the r, z $(\vec{i}, \vec{k})$ plane (Fig. 3), then the surface integral of P over the surface formed by rotating curve $(\ell, \ell+1)$ through an angle ϕ $(-\Theta_0, \Theta_0)$ about the z-axis is given by the line integrals

$$\int_{\ell}^{\ell+1} P d\vec{S} = \vec{i} \, \phi \int_{\ell}^{\ell+1} Prdz + \vec{k}\phi \int_{\ell+1}^{\ell} Prdr \quad . \tag{12}$$

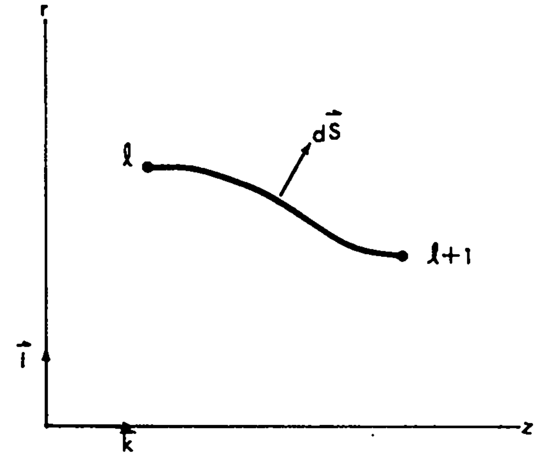

Fig. 3. Any curve (ℓ, ℓ+1) which is rotated through angle ϕ to give a surface

Proof: At any point on the surface formed

$$d\vec{S} = \vec{r}_1\,(rd\Theta dz) + \vec{z}_1\,(rd\Theta dr) \quad ,$$

so

$$\int_{\ell}^{\ell+1} Pd\vec{S} = \int_{-\Theta_0}^{\Theta} \int_{\ell}^{\ell+1} \vec{r}_1\,(Prd\Theta dz) + \int_{-\Theta_0}^{\Theta_0} \int_{\ell+1}^{\ell} \vec{z}_1\,(Prd\Theta dr) \quad .$$

Substituting from (10) and removing $\vec{i}$, $\vec{j}$, $\vec{k}$, from the integrals,

$$\int_{\ell}^{\ell+1} Pd\vec{S} = \vec{i} \int_{-\Theta_0}^{\Theta_0} \int_{\ell}^{\ell+1} Pr\cos\Theta\, d\Theta dz + \vec{j} \int_{-\Theta_0}^{\Theta_0} \int_{\ell}^{\ell+1} Pr\sin\Theta\, d\Theta dz$$

$$+ \vec{k} \int_{-\Theta_0}^{\Theta_0} \int_{\ell+1}^{\ell} Prd\Theta dr \quad .$$

Now, since P, r, dz, dr, are independent of Θ, we can remove them from the Θ integrals, giving

$$\int_{\ell}^{\ell+1} P d\vec{S} = \vec{i} \int_{z_\ell}^{z_{\ell+1}} \left(Pr \int_{-\Theta_0}^{\Theta_0} \cos\Theta \, d\Theta\right) dz + \vec{j} \int_{z_\ell}^{z_{\ell+1}} \left(Pr \int_{-\Theta_0}^{\Theta_0} \sin\Theta \, d\Theta\right) dz$$

$$+ \vec{k} \int_{r_{\ell+1}}^{r_\ell} \left(Pr \int_{-\Theta_0}^{\Theta_0} d\Theta\right) dr \quad .$$

Now, using (11) we get (12).

From now on, as mentioned in connection with Fig. 1, we will assume that P is constant in any zone, so we may remove P from the integrals and get from Theorem 1

Theorem 2: $$\int_{\ell}^{\ell+1} d\vec{S} = \vec{i}\phi \int_{\ell}^{\ell+1} r dz + \vec{k}\phi \int_{\ell+1}^{\ell} r dr$$

$$= \vec{i}\phi \int_{\ell}^{\ell+1} r dz + \vec{k}\phi \, \frac{r_\ell^2 - r_{\ell+1}^2}{2} \quad . \tag{13}$$

This says that the $\vec{k}$ (or z) integral is independent of the path the curve follows from ℓ to $\ell+1$, but the same is not true for the $\vec{i}$ (or r) integral.

Theorem 3. If the path from ℓ to $\ell+1$ is a straight line, then (13) becomes

$$\int_{\ell}^{\ell+1} d\vec{S} = \vec{i}\phi \left(z_{\ell+1} - z_\ell\right) \frac{r_{\ell+1} + r_\ell}{2} + \vec{k}\phi \left(r_\ell - r_{\ell+1}\right) \frac{r_{\ell+1} + r_\ell}{2} \quad . \tag{14}$$

The proof is straightforward if one takes $r = az + b$ and evaluates $\int_{\ell}^{\ell+1} r dz$.

Consider any triangle in the r, z $(\vec{i}, \vec{k})$ plane, with the vertices denoted 0, ℓ, $\ell+1$ as one goes around the triangle in a clockwise direction (Fig. 4).

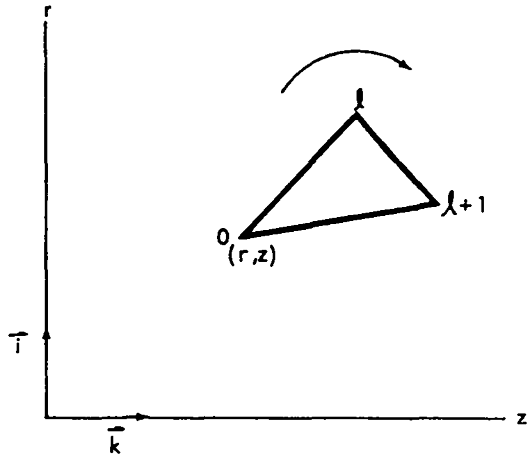

Fig. 4. A triangular zone.

<u>Theorem 4</u>. The area of the triangle is given by

$$A = \int_A drdz = \frac{1}{2}\left[(rz_\ell - r_\ell z) + (r_\ell z_{\ell+1} - r_{\ell+1} z_\ell) + (r_{\ell+1} z - r z_{\ell+1})\right]$$

$$= \frac{1}{2}\left[(z_{\ell+1} - z)\ (r_\ell - r) - (z_\ell - z)\ (r_{\ell+1} - r)\right] \quad . \tag{15}$$

The clockwise convention must be followed for the formulae in (15) to give a positive value for A. [A counter-clockwise convention would give negative areas with (15).] The proof is omitted.

Now consider the triangle of Fig. 4 to be rotated through an angle (from $-\Theta_0$ to Θ_0) about the z-axis to form a wedge-shaped volume centered about the $\vec{i}$ (or r) axis (Fig. 5).

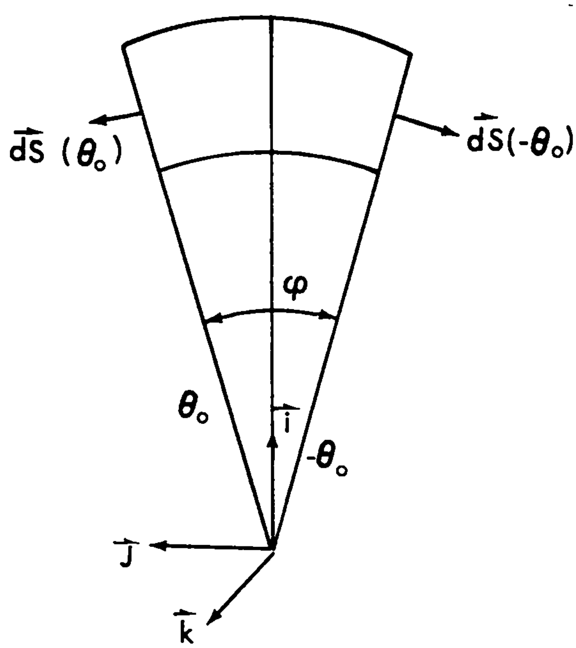

Fig. 5. End view of a wedge.

Theorem 5. The volume of this wedge is given by

$$V = \bar{r}\ A\phi \quad , \tag{16}$$

where

$$\bar{r} = \text{centroid of triangle} = \frac{1}{3}\left(r_0 + r_\ell + r_{\ell+1}\right) \quad .$$

Proof omitted.

Theorem 6. The sum of the surface integrals over both sides of the wedge formed by the rotated triangle (for small Θ_0) is given by

$$\oint_\phi d\vec{S} = \int d\vec{S}\ (-\Theta_0) + \int d\vec{S}\ (\theta_0) = -\ \vec{i}\ A\ \phi \quad , \tag{17}$$

where A, the area of the triangle, is given by (15).

Proof. From Fig. 5 it is obvious that the $\vec{j}$ components of the integrals cancel and that the $\vec{i}$ components add to give

$$\oint_{\phi} d\vec{S} = -\vec{i}2\left(A \sin \Theta_0\right) = -\vec{i}2A\Theta_0 = -\vec{i}\, A\, \phi \quad .$$

Now, under the assumption of constant P in any zone, it is apparent that the surface integrals over the side faces of the wedge and the surface integral over the surface formed by rotating the line ℓ, $\ell+1$ will have the same pressure, so it is possible to add these integrals. An interesting fact can be proved about this sum, namely that it is independent of the path from ℓ to $\ell+1$.

First let us derive the formula for this sum, which we denote by $\oint_{\ell}^{\ell+1} d\vec{S}$ (which is not the same as $\oint_{\phi} d\vec{S}$).

Theorem 7.

$$\oint_{\ell}^{\ell+1} d\vec{S} = \int_{\ell}^{\ell+1} d\vec{S} + \oint_{\phi} d\vec{S} = \vec{i}\,\frac{\phi}{2}\left[-z_\ell r_\ell - r(z_\ell - z_{\ell+1}) - z(r_{\ell+1} - r_\ell)\right.$$
$$\left. + z_{\ell+1}\; r_{\ell+1}\right] + \vec{k}\,\frac{\phi}{2}\left(r_\ell^2 - r_{\ell+1}^2\right) \quad . \qquad (18)$$

Proof. From (14) and (15) substituted in (17).

$$\oint_{\ell}^{\ell+1} d\vec{S} = \int_{\ell}^{\ell+1} d\vec{S} + \oint_{\phi} d\vec{S} = \vec{i}\,\frac{\phi}{2}\left[(z_{\ell+1} - z_\ell)\;(r_{\ell+1} + r_\ell)\right]$$
$$+ \vec{k}\,\frac{\phi}{2}\left[(r_\ell - r_{\ell+1})\;(r_{\ell+1} + r_\ell)\right] - \vec{i}\,\frac{\phi}{2}\left[(rz_\ell - r_\ell z)\right.$$
$$\left. + (r_\ell z_{\ell+1} - r_{\ell+1}\, z_\ell) + (r_{\ell+1}\, z - rz_{\ell+1})\right] \quad . \qquad (19)$$

Collecting terms and simplifying, we get (18).

<u>Theorem 8</u>. $\oint_{\ell}^{\ell+1} d\vec{S} = \int_{\ell}^{\ell+1} d\vec{S} + \oint_{\phi} d\vec{S}$ is independent of the path from ℓ to $\ell+1$.

<u>Proof</u>. Considering the triangle $(0, \ell, \ell+1)$, let us join the points ℓ to $\ell+1$ by a series of line segments $(j, j+1)$ where $j = 0, 1, 2, \ldots, J$. (Fig. 6.) Now join each j to 0, to form a series of triangles $(0, j, j+1)$.

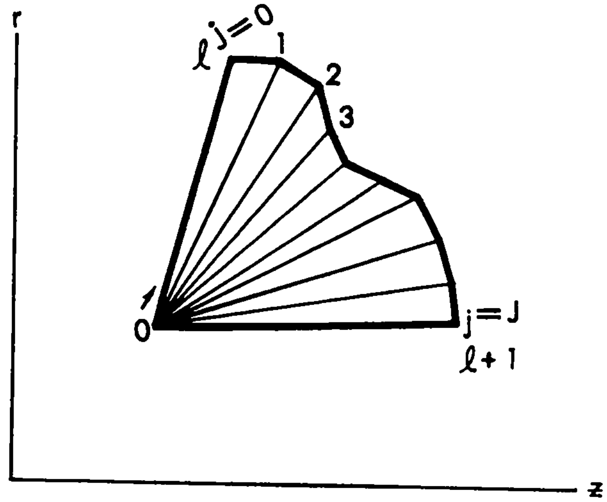

Fig. 6. Approximation of any curve ℓ, $\ell+1$ by a series of line segments between $j = 0, 1, 2, \ldots, J$.

Applying Theorem 7 to each of these triangles and summing, we have

$$\oint_{\ell}^{\ell+1} d\vec{S} = \sum_{j=0}^{J} \oint_{j}^{j+1} d\vec{S} = \vec{i}\,\frac{\phi}{2} \sum_{j=0}^{J} \left[-z_j\, r_j - r(z_j - z_{j+1})\right.$$

$$\left. - z\,(r_{j+1} - r_j) + z_{j+1}\, r_{j+1}\right] + \vec{k}\,\frac{\phi}{2} \sum_{j=0}^{J} (r_j^2 - r_{j+1}^2) \quad .$$

Because of cancellation of quantities from adjacent triangles, we have

$$\oint_{\ell}^{\ell+1} d\vec{S} = \vec{i}\,\frac{\phi}{2} [- z_{\ell}\, r_{\ell} - r\,(z_{\ell} - z_{\ell+1})$$

$$- z\,(r_{\ell+1} - r_{\ell}) + z_{\ell+1}\, r_{\ell+1}] + \vec{k}\,\frac{\phi}{2}\,(r_{\ell}^2 - r_{\ell+1}^2) \quad .$$

From this expression, we see that $\int_{\ell}^{\ell+1} d\vec{S}$ depends only on coordinates of points 0, ℓ, ℓ+1. Hence, if we let $J \to \infty$ we can approximate <u>any</u> curve between ℓ, ℓ+1 and therefore conclude that the value of the integral is independent of the path.

One can illustrate this theorem geometrically. The $\vec{i}$ term is given by the difference of two integrals, (13) and (15).

$$\int_{\ell}^{\ell+1} r dz - \int_A dr dz = (\text{Area under curve } \ell,\ \ell+1)$$

$$- (\text{Area enclosed by } 0,\ \ell,\ \ell+1) \quad .$$

From Fig. 7, we see that this difference remains constant as the path between (ℓ, ℓ+1) changes, for as the path is changed both areas are being changed by the same amount. For the $\vec{k}$ term, independence of path has already been shown in (13).

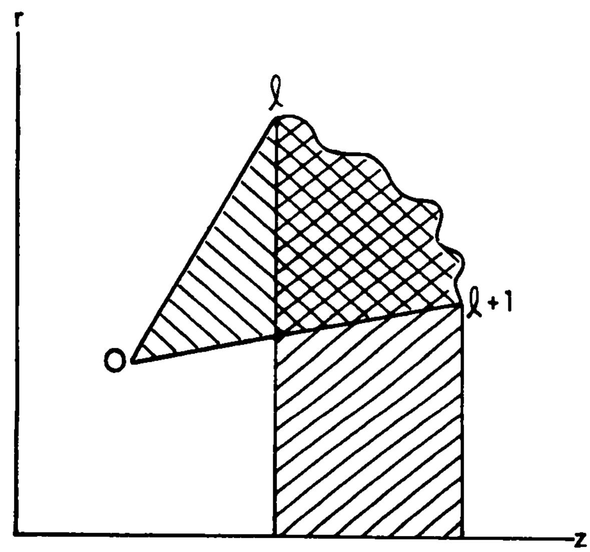

Fig. 7. Illustration of Theorem 8.

Now consider the total force on a typical zone of Fig. 1 (see Fig. 8).

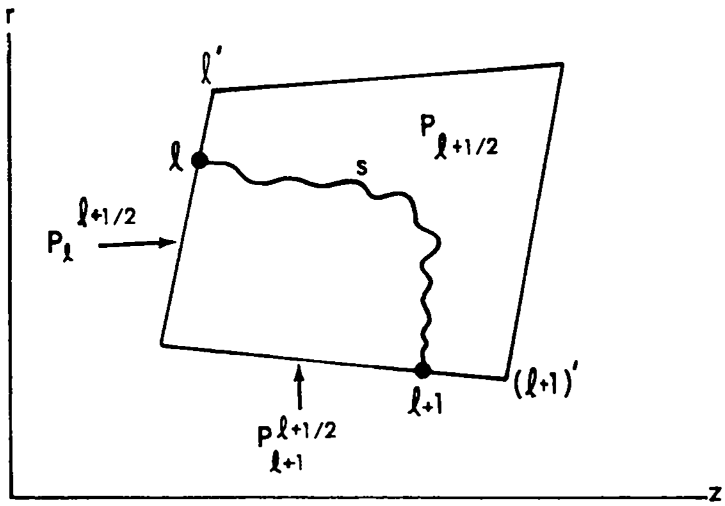

Fig. 8. The forces on a typical zone.

Theorem 9. Given points ℓ, $\ell+1$ along the boundaries, and any path S in the zone joining these points, and assuming constant pressures $P_{\ell}^{\ell+1/2}$, $P_{\ell+1}^{\ell+1/2}$ acting along these boundaries, and $P_{\ell+1/2}$ acting within the zone, then the total force acting on the material enclosed by S and the boundaries is given by

$$\vec{F}_{\ell+1/2} = -\left(\int_V P d\vec{S}\right)_{\ell+1/2} = -\left(P_{\ell}^{\ell+1/2} \int_0^{\ell} d\vec{S} + P_{\ell+1}^{\ell+1/2} \int_{\ell+1}^{0} d\vec{S}\right)$$

$$- P_{\ell+1/2}\left(\int_{\ell}^{\ell+1} d\vec{S} + \oint_{\phi} d\vec{S}\right) . \qquad (20)$$

This can be written out in detail, using (14) and (18), to give

$$\vec{F}_{\ell+1/2} = \vec{i}\ \frac{\phi}{2} \{P_{\ell}^{\ell+1/2} (z - z_{\ell})(r + r_{\ell}) + P_{\ell+1}^{\ell+1/2} (z_{\ell+1} - z)(r_{\ell+1} + r)$$

$$+ P_{\ell+1/2} [z_{\ell}\, r_{\ell} + r(z_{\ell} - z_{\ell+1}) + z(r_{\ell+1} - r_{\ell}) - z_{\ell+1}\, r_{\ell+1}]\}$$

$$+ \vec{k}\ \frac{\phi}{2}\left[P_{\ell}^{\ell+1/2}\,(r_{\ell} - r)(r + r_{\ell}) + P_{\ell+1}^{\ell+1/2}\,(r - r_{\ell+1})(r + r_{\ell+1})\right]$$

$$+ P_{\ell+1/2}\,(r_{\ell+1} - r_{\ell})(r_{\ell+1} + r_{\ell}) \quad . \tag{20a}$$

By algebraic manipulation this can be rewritten

$$\vec{F}_{\ell+1/2} = - \int_{V} P d\vec{S}$$

$$= \vec{i}\ \frac{\phi}{2}\left[(P_{\ell}^{\ell+1/2} - P_{\ell+1/2})(z - z_{\ell})(r + r_{\ell})\right.$$

$$\left. + (P_{\ell+1}^{\ell+1/2} - P_{\ell+1/2})(z_{\ell+1} - z)(r_{\ell+1} + r)\right]$$

$$+ \vec{k}\ \frac{\phi}{2}\left[(P_{\ell}^{\ell+1/2} - P_{\ell+1/2})(r_{\ell} - r)(r_{\ell} + r)\right.$$

$$\left. + (P_{\ell+1}^{\ell+1/2} - P_{\ell+1/2})(r - r_{\ell+1})(r + r_{\ell+1})\right] \quad . \tag{20b}$$

This is the <u>fundamental expression used to derive all gradients.</u>

It might be well to add a word about conservation of momentum. Since $\int P d\vec{S}$ is a momentum flux term, to conserve the momentum flow between two adjacent zones, say $\ell+1/2$ and $\ell-1/2$, then it is necessary to require that

$$\int_{0}^{\ell} P_{\ell}^{\ell+1/2}\, d\vec{S} = - \int_{\ell}^{0} P_{\ell}^{\ell-1/2}\, d\vec{S} \quad .$$

Since the pressures are assumed constant along $(0, \ell)$, this gives

$$P_{\ell}^{\ell+1/2} = P_{\ell}^{\ell-1/2} = P_{\ell} \quad . \tag{21}$$

We will use this principle to assure conservation of momentum and simplify notation.

We shall now determine how the general formulae (20a, 20b) simplify for the various gradients.

IGT: Using (20b) and (21) in the expression of $\sum_{\ell=1}^{L} \vec{F}_{\ell+1/2}$ in (7), we get

$$\sum_{\ell=1}^{L} \vec{F}_{\ell+1/2} = \vec{i}\,\frac{\phi}{2} \sum_{\ell=1}^{L} [(P_\ell - P_{\ell+1/2})(z - z_\ell)(r + r_\ell)$$

$$+ (P_{\ell+1} - P_{\ell+1/2})(z_{\ell+1} - z)(r_{\ell+1} + r)]$$

$$+ \vec{k}\,\frac{\phi}{2} \sum_{\ell=1}^{L} [(P_\ell - P_{\ell+1/2})(r_\ell - r)(r_\ell + r)$$

$$+ (P_{\ell+1} - P_{\ell+1/2})(r - r_{\ell+1})(r + r_{\ell+1})] \quad ,$$

in which we get cancellation of terms involving P_ℓ so that

$$\sum_{\ell=1}^{L} \vec{F}_{\ell+1/2} = \vec{i}\,\frac{\phi}{2} \sum_{\ell=1}^{L} [(P_{\ell-1/2} - P_{\ell+1/2})(z - z_\ell)(r + r_\ell)]$$

$$+ \vec{k}\,\frac{\phi}{2} \sum_{\ell=1}^{L} [(P_{\ell-1/2} - P_{\ell+1/2})(r_\ell - r)(r_\ell + r)] \quad . \tag{22}$$

IGA: There is no simplification possible here. See (8).

FGI: Referring to (9) and using (20b) and (21), we get with simplification,

$$\vec{F}_\ell = \vec{F}_{\ell-1/2} + \vec{F}_{\ell+1/2} = \vec{i}\,\frac{\phi}{2}\,[(P_{\ell-1} - P_{\ell-1/2})(z - z_{\ell-1})(r + r_{\ell-1})$$

$$+ (P_{\ell+1/2} - P_{\ell-1/2})(z_\ell - z)(r_\ell + r)$$

$$+ (P_{\ell+1} - P_{\ell+1/2})(z_{\ell+1} - z)(r_{\ell+1} + r)]$$

$$+ \vec{k}\,\frac{\phi}{2}\,[(P_{\ell-1} - P_{\ell-1/2})(r_{\ell-1} - r)(r_{\ell-1} + r)$$

$$+ (P_{\ell-1/2} - P_{\ell+1/2})(r_\ell - r)(r_\ell + r)$$

$$+ (P_{\ell+1} - P_{\ell+1/2})(r - r_{\ell+1})(r + r_{\ell+1})] \quad . \tag{23}$$

The formulae for FGI can be obtained from (23) by setting the first and last term in each component equal to zero. This is equivalent to the assumption

$$(r_{\ell-1}, z_{\ell-1}) = (r, z) \text{ and } (r_{\ell+1}, z_{\ell+1}) = (r, z) \quad , \tag{23a}$$

which gives for FGI

$$\vec{F}_\ell = \vec{i}\,\frac{\phi}{2}\,(P_{\ell+1/2} - P_{\ell-1/2})(z_\ell - z)(r_\ell + r)$$

$$+ \vec{k}\,\frac{\phi}{2}\,(P_{\ell-1/2} - P_{\ell+1/2})(r_\ell - r)(r_\ell + r) \quad . \tag{23b}$$

The assumption (23b) is quite interesting. It means, for example, that to get $\vec{F}_{\ell-1/2}$ used in $\vec{F}_\ell$, one uses a path 0 to ℓ completely in ℓ-1/2. Correspondingly, to get $\vec{F}_{\ell-1/2}$ used in $\vec{F}_{\ell-1}$, one uses a path ℓ-1 to 0 completely in ℓ-1/2. Since those paths have different end points, momentum is not conserved between these two $\int P d\vec{S}$ (Theorem 8).

If we take a longer view and consider the $\int P d\vec{S}$ for all four vertices 0, ℓ-1, ℓ", ℓ of a zone, we see that momentum can be conserved in total if all four paths pass through a common point 0' inside the zone, ℓ-1/2, and extend from vertex to vertex.

The momentum flux within the zone for each of the four corners is

$$0: \quad P\left(\int_0^{0'} d\vec{S} + \int_{0'}^{\ell} d\vec{S}\right) + P\left(\int_{\ell-1}^{0'} d\vec{S} + \int_{0'}^{0} d\vec{S}\right)$$

$$\ell-1:\quad P\left(\int_{\ell''}^{0'} d\vec{S} + \int_{0'}^{\ell-1} d\vec{S}\right) + P\left(\int_{\ell-1}^{0'} d\vec{S} + \int_{0'}^{0} d\vec{S}\right)$$

$$\ell'':\quad P\left(\int_{\ell}^{0'} d\vec{S} + \int_{0'}^{\ell''} d\vec{S}\right) + P\left(\int_{\ell''}^{0'} d\vec{S} + \int_{0'}^{\ell-1} d\vec{S}\right)$$

$$\ell:\quad P\left(\int_{0}^{0'} d\vec{S} + \int_{0'}^{\ell} d\vec{S}\right) + P\left(\int_{\ell}^{0'} d\vec{S} + \int_{0'}^{\ell''} d\vec{S}\right)$$

The net momentum flux, or total of all four expressions, is zero, so momentum is conserved among all four vertices for that zone. However, momentum is transferred diagonally as well as nondiagonally. This is not necessarily good or bad, but it makes visualization more difficult. The common point 0' does not appear in the formula (23a) for the forces, so it, along with the paths, is of use only in defining the masses used.

V. ANALYTICAL METHODS FOR MORE SPECIFIC DEFINITION OF INTEGRATED GRADIENTS

Since the ultimate objective in deriving the various types of integrated gradients defined in Sec. II is their use in numerical calculations of physical situations, it is true that comparison of numerical results with experimental observations is probably the best test of their validity. Indeed this is the method which led to adoption of FGI for the S. R. Orr code. However, there are certain idealized situations (planes, cylinders, spheres) in which one would like difference methods to give reasonably correct results. In the next few sections, we deal analytically with such motions. It has turned out that to achieve the desired behavior in some of these simple problems, certain consistent definitions of M, S, ℓ, etc., must be made. All of the following discussion will pertain to quadrilateral meshes. Some simple problems are:

<u>A Plane Problem</u>. This is a problem in which the material is divided into plane layers that are parallel to the r-axis, with uniform pressure, density, etc., in each layer. In other words, these quantities have no r dependence. The natural mesh to select is rectangular (Fig. 9). For such a problem, one

would hope that the accelerations in the $\vec{i}(r)$ direction vanish and that accelerations in the $\vec{k}(z)$ direction be independent of r.

A Cylindrical Problem. This is a problem in which there are cylindrical layers of material about the z-axis for which P, ρ, etc., are constant in each layer (that is, independent of z). The natural mesh for this type of problem is also rectangular (Fig. 10). For this situation one would expect that the accelerations in the $\vec{k}(z)$ direction vanish and that the accelerations in the $\vec{i}(r)$ direction be independent of z.

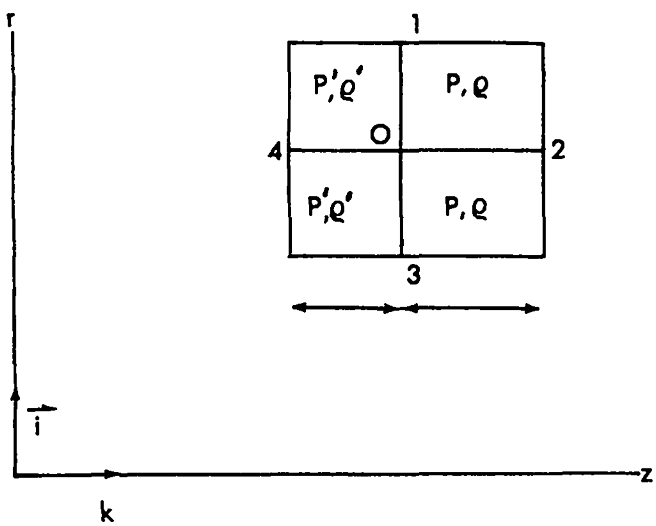

Fig. 9. A plane problem.

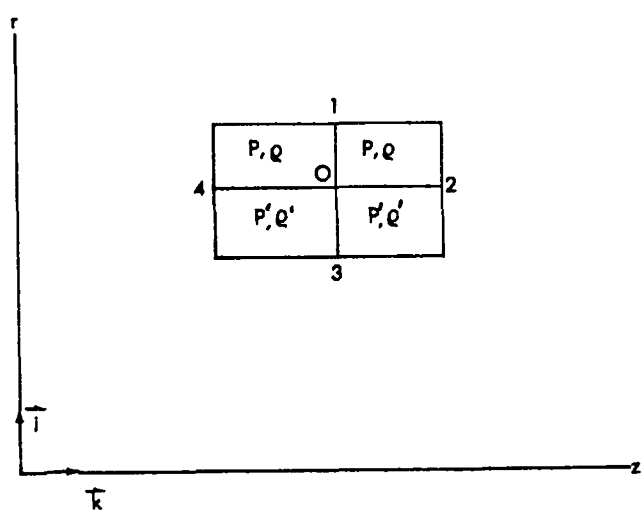

Fig. 10. A cylindrical problem.

A Spherical Problem. In this problem, the material is divided into spherical shells centered about the origin, with P, ρ, etc., constant for each shell (that is, independent of α). A natural mesh for this type of problem is quadrilateral (Fig. 11).

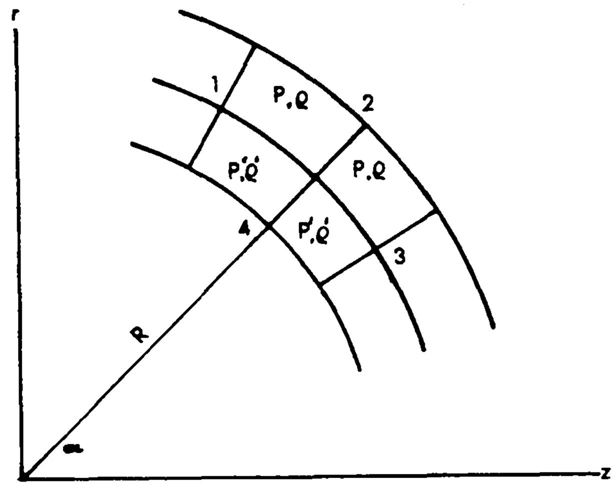

Fig. 11. A spherical problem.

For this problem, one would expect that the tangential (α) accelerations vanish, while the radial (R) accelerations be independent of α.

Limit Test. Another test to apply to the integrated gradients in difference form is to approximate the various quantities (P, ρ, etc.) in the difference equations by a Taylor Series expansion about the point 0, neglect higher order terms and see if the gradient approaches the proper differential form for that type of system. In other words, for:

plane problem $\qquad a_r \to 0,\ a_z \to -\frac{1}{\rho}\frac{\partial P}{\partial z}$

cylindrical problem $\qquad a_r \to -\frac{1}{\rho}\frac{\partial P}{\partial r},\ a_z \to 0$

spherical problem $\qquad a_R \to -\frac{1}{\rho}\frac{\partial P}{\partial R},\ a_\alpha \to 0 \quad . \qquad (24)$

VI. THE PLANE PROBLEM - q-MASS METHOD

As mentioned in (4) through (6), choices of ℓ and S (which define M) must be made. From the standpoint of computation, it is very advantageous to define $M_{\ell+1/2}$ as a constant fraction, q, of the mass, $m_{\ell+1/2}$, of the whole zone rotated through angle ϕ, that is,

$$M_{\ell+1/2} = q\phi m_{\ell+1/2} \quad . \qquad (25)$$

This means that M need be calculated only once, at the start of the problem, for each zone. We have named this general approach "the q method." Based on this general assumption, the plane problem can be used to indicate a logical choice for ℓ. For the plane problem (Fig. 12), we can use Theorem 5 (or other methods) to get the masses $M_{\ell+1/2}$. (Here, as in much of the following work, detailed algebraic steps will be omitted to conserve space.)

$$M_{1+1/2} = q\phi m_{1+1/2} = q \frac{\phi}{2} \rho \, \Delta r_1 \, \Delta z_2 \, (2r + \Delta r_1)$$

$$M_{2+1/2} = q\phi m_{2+1/2} = q \frac{\phi}{2} \rho \, \Delta r_3 \, \Delta z_2 \, (2r - \Delta r_3)$$

$$M_{3+1/2} = q\phi m_{3+1/2} = q \frac{\phi}{2} \rho' \, \Delta r_3 \, \Delta z_4 \, (2r - \Delta r_3)$$

$$M_{4+1/2} = q\phi m_{4+1/2} = q \frac{\phi}{2} \rho' \, \Delta r_1 \, \Delta z_4 \, (2r + \Delta r_1)$$

$$\sum_{\ell=1}^{4} M_{\ell+(1/2)} = q \frac{\phi}{2} \left[\Delta r_1 \, (2r + \Delta r_1) + \Delta r_3 \, (2r - \Delta r_3)\right] \left(\rho' \Delta z_4 + \rho \Delta z_2\right) \quad . \tag{26}$$

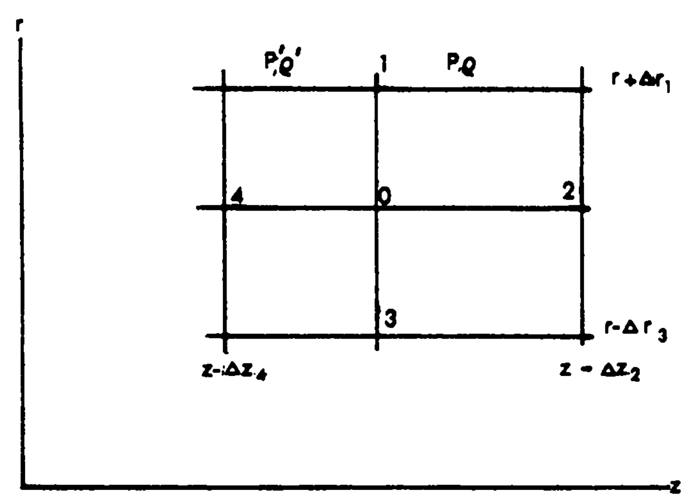

Fig. 12. The q-method in a plane problem.

To determine ℓ, the intersection of S with the common boundary between zones $\ell-1/2$, $\ell+1/2$ (Fig. 1), assume ℓ to be located at some fraction, f, of the total distance from 0 out to the next vertex. For example, for $\ell = 1$,

$$\frac{r_1 - r}{r_1' - r} = f \quad \text{or } r_1 = r + f\Delta r_1$$

$$z_1 = z + f\Delta z_1 \quad . \tag{27}$$

There is one more logical assumption we can make for the plane problem. Since $P_{1+1/2} = P_{2+1/2} = P$ and $P_{3+1/2} = P_{4+1/2} = P'$, it seems imperative that here we define P_ℓ so that

$$P_2 = P,\ P_4 = P',\ P_1 = P_3 \quad . \tag{28}$$

This will simplify many of the formulae.

We now consider the various gradients:

<u>IGT-q</u>: Applying (22) to Fig. 12, in conjunction with (27).

$$\sum_{\ell=1}^{4} \vec{F}_{\ell+1/2} = \vec{i}\,\frac{\phi}{2}\left[(P' - P)(0)(r + r_1) + 0 + (P - P')(0)(r + r_3) + 0\right]$$

$$+ \vec{k}\,\frac{\phi}{2}\left[(P' - P)(f\Delta r_1)(2r + f\Delta r_1) + 0 + (P - P')(-f\Delta r_3)(2r - f\Delta r_3) + 0\right]$$

$$= \vec{i}(0) + \vec{k}\;\frac{\phi}{2}\,(P' - P)f\left[\Delta r_1(2r + f\Delta r_1) + \Delta r_3(2r - f\Delta r_3)\right] \quad . \tag{29}$$

Substituting (29), (26) in (7) ,

$$\vec{a} = \frac{\sum \vec{F}_{\ell+1/2}}{\sum M_{\ell+1/2}} = \frac{\vec{i}(0) + \vec{k}\;\frac{\phi}{2}\,(P' - P)f\left[\Delta r_1(2r + f\Delta r_1) + \Delta r_3(2r - f\Delta r_3)\right]}{q\;\frac{\phi}{2}\,(\rho'\Delta z_4 + \rho\Delta z_2)\left[\Delta r_1(2r + \Delta r_1) + \Delta r_3\,(2r - \Delta r_3)\right]} \; .$$

From this we see that $a_r = 0$, as it should in a plane problem and that a_z will be independent of r and Δr if <u>$f = 1$</u>. In this case

$$a_z = \frac{(P' - P)}{q\,(\rho'\Delta z_4 + \rho\Delta z_2)} \quad . \tag{30}$$

Applying the limit test (24), and dropping out second order terms

$$a_z = \frac{\left(P_0 - \frac{\partial P}{\partial z}\frac{\Delta z_4}{2}\right) - \left(P_0 + \frac{\partial P}{\partial z}\frac{\Delta z_2}{2}\right)}{q\left[\left(\rho_0 - \frac{\partial \rho}{\partial z}\frac{\Delta z_4}{2}\right)\Delta z_4 + \left(\rho_0 + \frac{\partial \rho}{\partial z}\frac{\Delta z_2}{2}\right)\Delta z_2\right]}$$

$$a_z \to \frac{-\frac{\partial P}{\partial z}\,\frac{1}{2}\,(\Delta z_4 + \Delta z_2)}{q\,\rho_0\,(\Delta z_4 + \Delta z_2)} \quad ,$$

or

$$a_z \to -\frac{1}{2q\rho_0}\frac{\partial P}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} \text{ if } q = \frac{1}{2} \quad .$$

The conclusion is that for IGT-q to work properly in a plane problem, it is necessary to take: $q = 1/2$, $f = 1$.

<u>IGA-q</u>: Using (27), (28) in (20b) for Fig. 12,

$$\vec{F}_{1+1/2} = \vec{i}(0) + \vec{k}\,\frac{\phi}{2}\left[(P_1 - P)(f\Delta r_1)(2r + f\Delta r_1) + 0\right]$$

$$\vec{F}_{2+1/2} = \vec{i}(0) + \vec{k}\,\frac{\phi}{2}\left[0 + (P_3 - P)(f\Delta r_3)(2r - f\Delta r_3)\right]$$

$$\vec{F}_{3+1/2} = \vec{i}(0) + \vec{k}\,\frac{\phi}{2}\left[(P_3 - P')(-f\Delta r_3)(2r - f\Delta r_3) + 0\right]$$

$$\vec{F}_{4+1/2} = \vec{i}(0) + \vec{k}\,\frac{\phi}{2}\left[0 + (P_1 - P')(-f\Delta r_1)(2r + f\Delta r_1)\right] \quad . \tag{31}$$

Substitute these and (26) in (8) using (28),

$$\vec{a} = \frac{1}{4} \sum \frac{\vec{F}_{\ell+1/2}}{M_{\ell+1/2}} = \vec{i}(0) + \vec{k} \left\{ \frac{1}{4q} \left(\frac{P_1 - P}{\rho \Delta z_2} + \frac{P' - P_1}{\rho' \Delta z_4} \right) \left[\frac{f(2r + f\Delta r_1)}{(2r + \Delta r_1)} + \frac{f(2r - f\Delta r_3)}{(2r - \Delta r_3)} \right] \right\} .$$

From this expression, we see that $a_r = 0$ and a_z is independent of r and Δr <u>if $f = 1$</u>. In this case,

$$a_z = \frac{1}{2q} \left(\frac{P_1 - P}{\rho \Delta z_2} + \frac{P' - P_1}{\rho' \Delta z_4} \right), \tag{32}$$

with P_1 as yet undefined. We have arbitrarily chosen the simplest way to define P_ℓ which agrees with (28), namely

$$P_\ell = \frac{P_{\ell-1/2} + P_{\ell+1/2}}{2} \text{ or } P_1 = \frac{P' + P}{2} . \tag{33}$$

This gives for (32)

$$a_z = \left(\frac{P' - P}{4q} \right) \left(\frac{1}{\rho \Delta z_2} + \frac{1}{\rho' \Delta z_4} \right) ,$$

for which the limit test gives (for equal Δz)

$$a_z = \left(\frac{P' - P}{4q} \right) \left(\frac{2}{\rho \Delta z} \right) \rightarrow - \frac{1}{\rho} \frac{\partial P}{\partial z} \quad \text{if } q = 1/2 .$$

The conclusion is that for IGA-q to give proper motion in a plane problem it is best to use $q = 1/2$, $f = 1$.

<u>FGI-q</u>. Substituting (27), (28) in (23b) for Fig. 12

$$\vec{F}_1 = \vec{i}(0) + \vec{k} \frac{\phi}{2} \left[(P' - P)(f\Delta r_1)(2r + f\Delta r_1) \right]$$

$$\vec{F}_2 = \vec{i}(0) + \vec{k}\,\frac{\phi}{2}\,(0)$$

$$\vec{F}_3 = \vec{i}(0) + \vec{k}\,\frac{\phi}{2}\,[(P - P')(-f\Delta r_3)(2r - f\Delta r_3)]$$

$$\vec{F}_4 = \vec{i}(0) + \vec{k}\,\frac{\phi}{2}\,(0) \quad . \tag{34}$$

Now using (34) with (26) in (9),

$$\vec{a} = \vec{i}(0) + \frac{\vec{k}}{4q}\left[\frac{(P' - P)(f\Delta r_1)(2r + f\Delta r_1)}{\Delta r_1(2r + \Delta r_1)(\rho'\Delta z_4 + \rho\Delta z_2)} + \frac{(P' - P)(f\Delta r_3)(2r - f\Delta r_3)}{\Delta r_3(2r - \Delta r_3)(\rho'\Delta z_4 + \rho\Delta z_2)}\right]$$

$$= \vec{k}\left\{\frac{(P' - P)}{4q(\rho'\Delta z_4 + \rho\Delta z_2)}\left[\frac{f(2r + f\Delta r_1)}{2r + \Delta r_1} + \frac{f(2r - f\Delta r_3)}{2r - \Delta r_3}\right]\right\} .$$

From this expression, we see that $a_r = 0$ and a_z will be independent of r, Δr if $\underline{f = 1}$, in which case

$$a_z = \frac{P' - P}{2q(\rho'\Delta z_4 + \rho\Delta z_2)} \quad . \tag{35}$$

Applying the limit test,

$$a_z \to -\frac{1}{4q\rho}\,\frac{\partial P}{\partial z} = -\frac{1}{\rho}\,\frac{\partial P}{\partial z} \quad \text{if } q = 1/4 \quad .$$

Our conclusion is that for FGI-q to work properly in the plane problem it is best to use $f = 1$, $q = 1/4$.

<u>Summary and Discussion</u>. In the plane problem, all three methods (IGT, IGA, and FGI) give $a_r = 0$ as desired. It is possible to achieve a_z independent of r and Δr if we take $f = 1$ for all three gradients. Then the value q required to give a proper limit, namely $-\frac{1}{\rho}\frac{\partial P}{\partial z}$, for the gradients is:

IGT-q q = 1/2

IGA-q q = 1/2

FG1-q q = 1/4

These values of f and q lead to a physical model for each gradient. By this we mean a physical visualization of the mass M associated with each vertex and a surface S which encloses it. With this model it is possible to consider how momentum is transferred from one point to another by the gradient in question.

In IGT-q where f = 1, q = 1/2, we have a model (Fig. 13a) in which ℓ is taken out to the next vertex along the side (because f = 1) and S is drawn in any way between the proper end points so as to enclose half the mass of the zone (since q = 1/2). [Recall that $\int P d\vec{S}$ is independent of the shape of S in the zone (Theorem 8), so any curve between the proper end points on sides ℓ, $\ell+1$ that encloses the proper mass will serve as a representation.] Similarly, for IGA-q (Fig. 13b) we show surfaces S, which extend from the end points of the sides (f = 1) and enclose half the mass of the zones (q = 1/2). FGI-q is less obvious (Fig. 13c). This method, according to (9), adds

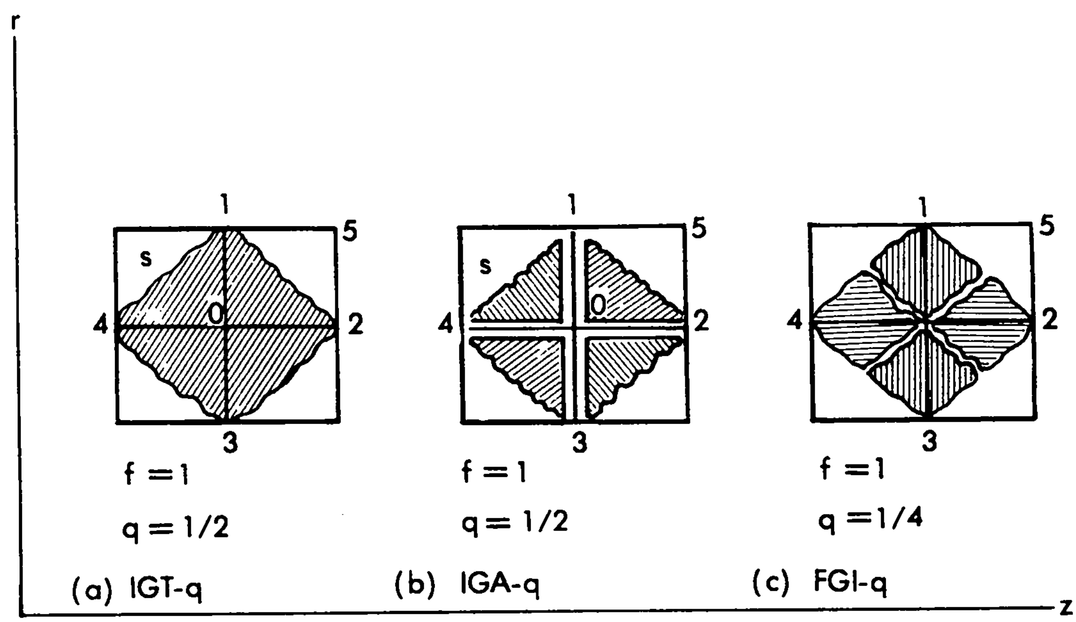

Fig. 13. Models for the various q-gradients.

surface integrals from two adjacent zones, and then by (23a) brings in the end points at $\ell-1$ and $\ell+1$ to point 0, thereby giving what may be represented by a diamond-shaped mass straddling each of the four sides. This diamond-shaped mass extends out to the end of the side ($f = 1$) and encloses one-fourth of the mass of each zone ($q = 1/4$).

It is interesting and instructive to consider how these models (and hence the corresponding equations) handle the conservation of momentum. Since $\int P d\vec{S}$ represents a rate of transfer of momentum, we can analyze momentum conservation in a mesh by considering the relationships of the $\int P d\vec{S}$ of adjacent vertices. For the q method, in general, momentum is conserved between vertices of points which diagonally oppose each other across a zone. For example, in IGT-q (Fig. 13a) when working on point 0, one uses $\int_1^2 P d\vec{S}$ for zone 1+1/2. Similarly, when working on point 5, one uses $\int_2^1 P d\vec{S}$ for zone 1+1/2. Since the endpoints of two integrals are the same because $f = 1$, the two integrals will be equal in magnitude but opposite in sign, which means that one point gains the momentum that the other loses, and hence momentum is conserved in the problem as a whole. For IGA-q and FGI-q, similar arguments hold. In addition, by similar arguments there is momentum conservation between the adjacent $M_{\ell+1/2}$ used within these gradients.

One objection to all the q methods is that the masses over which one integrates for adjacent points overlap each other. This objection does not seem vital for the gradients, but it leads to almost insurmountable difficulties when one tries to use the models for accomplishing rezoning or viscosity.[1,2,3] In the next section, we propose a method for defining masses which does not have this overlapping of masses.

VII. THE PLANE PROBLEM, MAC-0 MASS METHOD (Midpoint, Average Centroid Method)

Consider zone 1+1/2 of Fig. 12, and draw any set of curves joining the midpoints of adjacent sides (Fig. 14a). These curves define surfaces such that there will be complete conservation of momentum among the four vertices of this zone because of $\int P d\vec{S}$. This is because $\int P d\vec{S}$ is independent of the path

(Theorem 8), and hence the integral about the closed path vanishes. Next, one may vary these curves to define almost any masses desired without affecting the momentum conservation. However, we wish to use the total mass of the zone and at the same time avoid the overlapping of the masses associated with the vertices. One simple way to

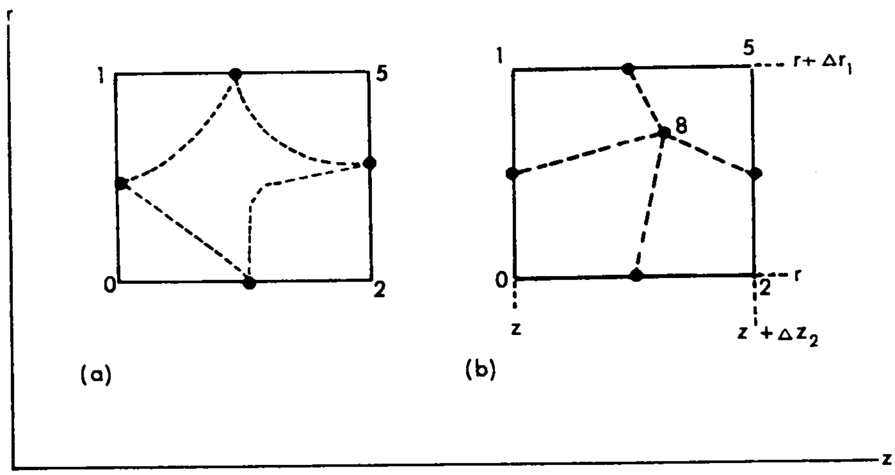

Fig. 14. The MAC-0 method.

accomplish this is to select some point 8 $(r_{1+1/2}, z_{1+1/2})$ within the zone and make all the curves pass through this point. For our discussion we will assume that the curves joining the midpoints to point 8 are straight lines (Fig. 14b). If we define point 8 with the two parameters g, h, by

$$r_{1+1/2} = r + g\Delta r_1 \qquad\qquad z_{1+1/2} = z + h\Delta z_2 \quad ,$$

then the part of the mass in zone 1+1/2 which is associated with point 0 is given by Theorem 5 as

$$M_{1+1/2} = \rho\,\phi\left\{\left[\frac{1}{2}\left(\frac{\Delta r_1}{2}\right)\left(h\Delta z_2\right)\right]\left[r + \frac{\Delta r_1}{3}\left(\frac{1}{2} + g\right)\right] + \left[\frac{1}{2}\left(\frac{\Delta z_2}{2}\right)\left(g\Delta r_1\right)\right]\left(r + \frac{\Delta r_1}{3}\,g\right)\right\}$$

$$= \rho\,\phi\,\frac{\Delta r_1 \Delta z_2}{4}\left[r(h + g) + \frac{\Delta r_1}{3}\left(\frac{h}{2} + hg + g^2\right)\right] .$$

In defining the model in Fig. 14b, we have used the midpoints of the sides, which means $f = 1/2$. Looking at the expressions for $\vec{F}_{1+1/2}$ in (29), (31), (34), we see that they all have factors like $[2r + (\Delta r_1)/2] = 2[r + (\Delta r_1)/4]$, etc. To achieve independence of r, we also want similar factors in $M_{1+1/2}$. The obvious way to get terms like this is to take

$$g = h = 1/2 \tag{36}$$

For this simple rectangular zone there are a number of possible interpretations of (36). For example, we could define point 8 as

(a) The intersection of the diagonals of the zone

(b) The average centroid of the zone, that is,

$$r_{1+1/2} = \frac{1}{4}\left(r + r_1 + r_5 + r_2\right) \qquad z_{1+1/2} = \frac{1}{4}\left(z + z_1 + z_5 + z_2\right) \tag{37}$$

(c) The real centroid of the zone, that is, dividing the zone into two triangles denoted by A, B, and calculating (proof omitted)

$$r_{1+1/2} = \frac{\bar{r}^A_{1+1/2}\, A^A_{1+1/2} + \bar{r}^B_{1+1/2}\, A^B_{1+1/2}}{A^A_{1+1/2} + A^B_{1+1/2}}$$

$$z_{1+1/2} = \frac{\bar{z}^A_{1+1/2}\, A^A_{1+1/2} + \bar{z}^B_{1+1/2}\, A^B_{1+1/2}}{A^A_{1+1/2} + A^B_{1+1/2}}\ , \tag{38}$$

where A^A means the area of triangle A and A^B means the area of triangle B.

Note that the centroids are not the same quantities as the centers of mass for the wedge-shaped volume formed by the rotation through angle ϕ. There may be other interpretations of (36). In later sections, when discussing other types of problems, it will be shown that the average centroid method, (37), seems preferable to use in general.

If we calculate $M_{\ell+1/2}$ for Fig. 14 under assumptions (36), (37),

$$M_{1+1/2} = \rho \frac{\phi}{2} \frac{\Delta r_1 \Delta z_2}{4} \left(2r + \frac{\Delta r_1}{2}\right)$$

$$M_{2+1/2} = \rho \frac{\phi}{2} \frac{\Delta r_3 \Delta z_2}{4} \left(2r - \frac{\Delta r_3}{2}\right)$$

$$M_{3+1/2} = \rho' \frac{\phi}{2} \frac{\Delta r_3 \Delta z_4}{4} \left(2r - \frac{\Delta r_3}{2}\right)$$

$$M_{4+1/2} = \rho' \frac{\phi}{2} \frac{\Delta r_1 \Delta z_4}{4} \left(2r + \frac{\Delta r_1}{2}\right) \quad . \tag{39}$$

We now apply (39) to the various gradients.

<u>IGT</u>. Using f = 1/2 in (29), and (39),

$$\vec{a} = \frac{\sum \vec{F}_{\ell+1/2}}{\sum M_{\ell+1/2}} = \frac{\vec{i}(0) + \vec{k}\frac{\phi}{2}(P' - P) \ \frac{1}{2}\left[\Delta r_1\left(2r + \frac{\Delta r_1}{2}\right) + \Delta r_3\left(2r - \frac{\Delta r_3}{2}\right)\right]}{\frac{\phi}{2}\frac{1}{4}\left(\rho\Delta z_2 + \rho'\Delta z_4\right) \ \left[\Delta r_1\left(2r + \frac{\Delta r_1}{2}\right) + \Delta r_3\left(2r - \frac{\Delta r_3}{2}\right)\right]} \quad .$$

whence

$$a_r = 0 \quad a_z = \frac{P' - P}{\frac{1}{2}\left(\rho'\Delta z_4 + \rho\Delta z_2\right)} \rightarrow -\frac{1}{\rho}\frac{\partial P}{\partial z} \quad , \tag{40}$$

which is independent of r and Δr.

<u>IGA</u>. Using f = 1/2 in (31), and (39),

$$\vec{a} = \frac{1}{4}\ \frac{\sum \vec{F}_{\ell+1/2}}{\sum M_{\ell+1/2}} = \vec{i}(0) + \frac{\vec{k}\frac{1}{4}\frac{\phi}{2}\frac{1}{2}}{\frac{1}{4}\frac{\phi}{2}} \left[\frac{\left(P_1 - P\right) + \left(P_3 - P\right)}{\rho\ \Delta z_2}\right.$$

$$+ \frac{(P' - P_3) + (P' - P_1)}{\rho' \Delta z_4} \Bigg] .$$

Thus $a_r = 0$ and using (28), that is, $P_1 = P_3$,

$$a_z = \frac{P_1 - P}{\rho \Delta z_2} + \frac{P' - P_1}{\rho' \Delta z_4} \tag{41}$$

and (33)

$$a_z = \frac{P' - P}{2} \left(\frac{1}{\rho \Delta z_2} + \frac{1}{\rho' \Delta z_4} \right) \rightarrow - \frac{1}{\rho} \frac{\partial P}{\partial z} .$$

FGI. In this instance, we must use (1/2) $M_{\ell+1/2}$ from (39) to achieve the proper limit. At first this seems strange, but if we visualize the masses that are needed for the ℓth term in FGI (Fig. 15), we see that to avoid overlapping of the masses used, it is necessary to use (1/2) $M_{\ell-1/2}$ and (1/2) $M_{\ell+1/2}$. Now, we use f = 1/2 in (34) and (1/2)M from (39). This gives

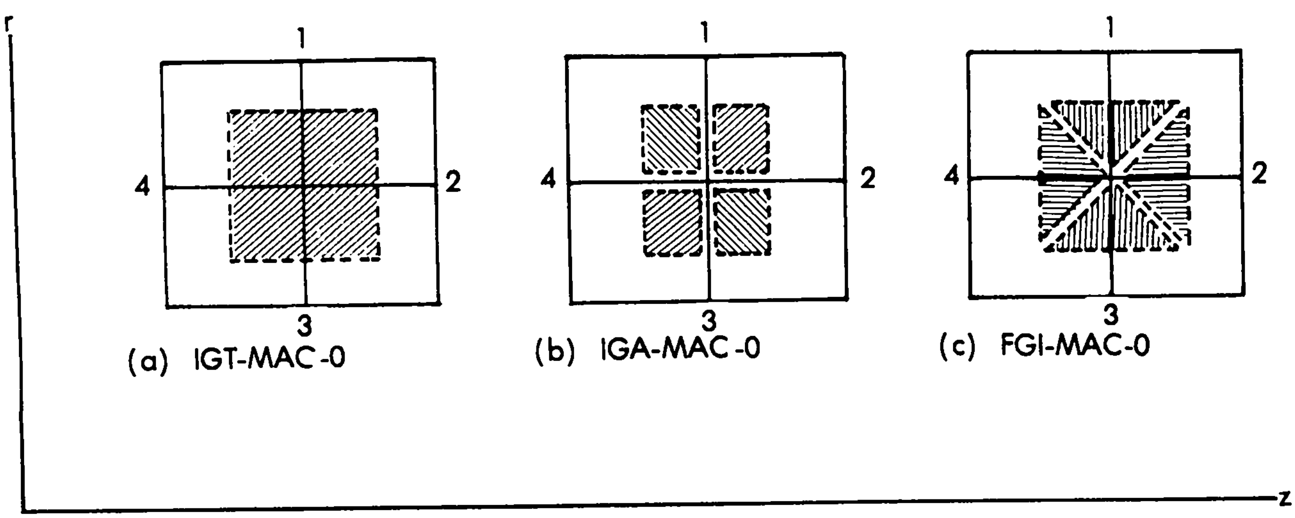

Fig. 15. Representation of MAC-0 masses in all three gradients.

$$\vec{a} = \frac{1}{4} \sum \frac{\vec{F}_{\ell-1/2} + \vec{F}_{\ell+1/2}}{M_{\ell-1/2} + M_{\ell+1/2}}$$

$$= \vec{i}(0) + \frac{\vec{k}\,\frac{1}{4}\,\frac{\phi}{2}\,\frac{1}{2}}{\frac{1}{2}\,\frac{\phi}{2}\,\frac{1}{4}} \left[\frac{(P' - P)}{\rho\Delta z_2 + \rho'\Delta z_4} + \frac{(P' - P)}{\rho\Delta z_2 + \rho'\Delta z_4} \right] .$$

So $a_r = 0$ and

$$a_z = \frac{2(P' - P)}{\rho\Delta z_2 + \rho'\Delta z_4} \rightarrow -\frac{1}{\rho}\frac{\partial P}{\partial z} . \tag{42}$$

Summary. Here again, $a_r = 0$ regardless of the assumptions. If we define the masses, $M_{\ell+1/2}$, as those enclosed by joining the midpoints of the sides ($f = 1/2$) to the centroid of the zone, all three methods give an a_z independent of r, Δr. This a_z approaches the proper limit, $-\frac{1}{\rho}\frac{\partial P}{\partial z}$, as the spacing becomes small. (By the nature of the definition of FGI, for it we must use half the masses of the subzones.)

All methods conserve momentum between adjacent points rather than merely diagonally as in the q-method for defining masses. This seems more physically realistic. In addition, this method also uses the more appealing concept of having no overlapping of the masses between points or within points in setting up the definitions.

VIII. THE CYLINDRICAL PROBLEM - q-MASS METHOD

For a typical vertex in the cylindrical problem (Fig. 16),

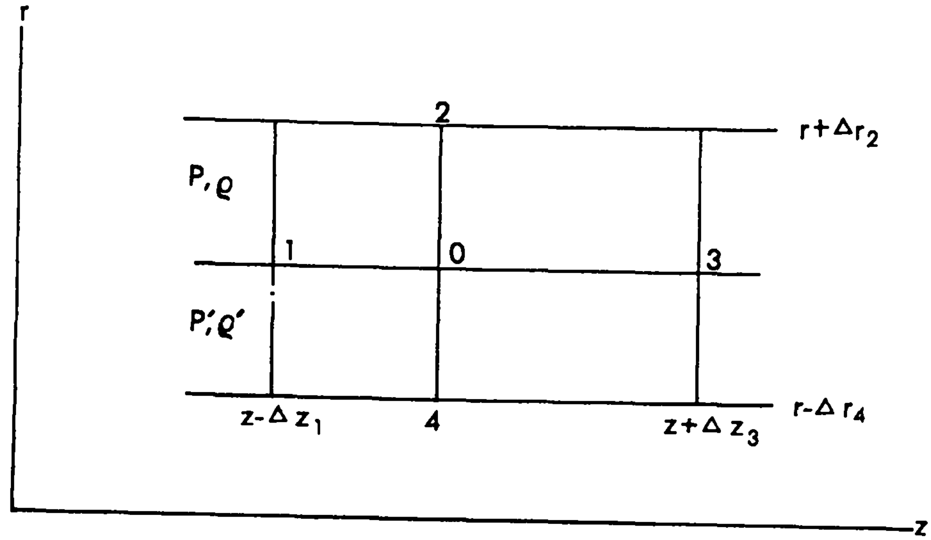

Fig. 16. The q-method in a cylindrical problem.

use of theorem 5 gives the masses of the zones as

$$M_{1+1/2} = q\phi\rho\ (1/2)(\Delta z_1 \Delta r_2)\ (2r + \Delta r_2)$$

$$M_{2+1/2} = q\phi\rho\ (1/2)(\Delta z_3 \Delta r_2)\ (2r + \Delta r_2)$$

$$M_{3+1/2} = q\phi\rho'\ (1/2)(\Delta z_3 \Delta r_4)\ (2r - \Delta r_4)$$

$$M_{4+1/2} = q\phi\rho'\ (1/2)(\Delta z_1 \Delta r_4)\ (2r - \Delta r_4) \quad . \qquad (43)$$

Now apply these to the various gradients with f [see (27)] unknown.

<u>IGT-q</u>: From (22) using (27):

$$\sum_{\ell=1}^{4} \vec{F}_{\ell+1/2} = \vec{i}\ \frac{\phi}{2}\ [(P' - P)(f\Delta z_1)(2r) + 0 + (P - P')(-f\Delta z_3)\ (2r) + 0]$$

$$+ \vec{k}\ \frac{\phi}{2}\ (0 + 0 + 0 + 0)$$

$$= \vec{i}\,\frac{\phi}{2}\,[(P' - P)2rf(\Delta z_1 + \Delta z_3) + \vec{k}(0) \quad . \tag{44}$$

Hence from (43), (7),

$$\vec{a} = \frac{\sum \vec{F}}{\sum M} = \frac{\vec{i}\,\frac{\phi}{2}\,(P' - P)2rf(\Delta z_1 + \Delta z_3)}{q\,\frac{\phi}{2}\,(\Delta z_1 + \Delta z_3)[\rho \Delta r_2(2r + \Delta r_2) + \rho' \Delta r_4(2r - \Delta r_4)]} \quad ,$$

so

$$a_z = 0 \quad a_r = \frac{f(P' - P)}{q[(\rho \Delta r_2 + \rho' \Delta r_4) + \frac{\rho \Delta r_2^2 - \rho' \Delta r_4^2}{2r}]} \quad . \tag{45}$$

Thus, we find that $a_z = 0$ and a_r is independent of z and Δz regardless of the choice of f. In the limit

$$a_r \rightarrow -\frac{f}{2q}\,\frac{1}{\rho}\,\frac{\partial P}{\partial r} \quad .$$

This approaches the proper value $-\frac{1}{\rho}\frac{\partial P}{\partial r}$ if $\frac{f}{q} = 2$. As in the plane problem $q = 1/2$, $f = 1$ would be suitable choice.

The conclusion is that for IGT-q to work properly in a cylindrical problem, it is sufficient to use $f/q = 2$.

IGA-q: Using (43) and (28) in (20b),

$$\vec{F}_{1+1/2} = \vec{i}\,\frac{\phi}{2}\,[(P_1 - P)(f\Delta z_1)(2r) + 0] + \vec{k}(0)$$

$$\vec{F}_{2+1/2} = \vec{i}\,\frac{\phi}{2}\,[0 + (P_3 - P)(f\Delta z_3)(2r)] + \vec{k}(0)$$

$$\vec{F}_{3+1/2} = \vec{i}\,\frac{\phi}{2}\,[(P_3 - P')(-f\Delta z_3)(2r) + 0] + \vec{k}(0)$$

$$\vec{F}_{4+1/2} = \vec{i}\,\frac{\phi}{2}\left[0 + (P_1 - P')(-f\Delta z_1)(2r)\right] + \vec{k}(0) \quad . \tag{46}$$

Using (46), (43), and (28) in (8),

$$\vec{a} = \frac{1}{4}\sum\left(\frac{\vec{F}}{M}\right)_{\ell+1/2} = \vec{i}\,\frac{f}{2q}\left[\frac{P_1 - P}{\rho\Delta r_2\left(1 + \frac{\Delta r_2}{2r}\right)} + \frac{P' - P_1}{\rho'\Delta r_4\left(1 - \frac{\Delta r_4}{2r}\right)}\right] + \vec{k}(0) \quad . \tag{47}$$

Thus, $a_z = 0$ and a_r is independent of z and Δz. If we define P_1 by (33) and apply the limit test,

$$a_r \rightarrow -\frac{f}{2q}\,\frac{1}{\rho}\,\frac{\partial P}{\partial r} \quad ,$$

which gives the proper value $-\frac{1}{\rho}\frac{\partial P}{\partial r}$ if $\frac{f}{q} = 2$, as in the plane problem. Here again we conclude that for IGA-q to work properly in the cylindrical problem, it is sufficient to use $f = 1$, $q = 1/2$.

FGI-q. Applying (23b) to Fig. 17,

$$\vec{F}_1 = \vec{i}\,\frac{\phi}{2}\,(P' - P)(f\Delta z_1)(2r) + \vec{k}(0)$$

$$\vec{F}_2 = \vec{i}(0) + \vec{k}(0)$$

$$\vec{F}_3 = \vec{i}\,\frac{\phi}{2}\,(P' - P)(f\Delta z_3)(2r) + \vec{k}(0)$$

$$\vec{F}_4 = \vec{i}(0) + \vec{k}(0) \quad . \tag{48}$$

Using (48) and (43) in (9),

$$\vec{a} = \frac{1}{4} \sum_{\ell} \left(\frac{\vec{F}_{\ell}}{M_{\ell-1/2} + M_{\ell+1/2}} \right)$$

$$= \vec{i}\ \frac{f}{2q} \left[\frac{(P' - \quad P)}{(\rho \Delta r_2 + \rho' \Delta r_4) + \left(\dfrac{\rho \Delta r_2^2 - \rho' \Delta r_4^2}{2r} \right)} \right] + \vec{k}(0) \ . \qquad (49)$$

Hence $a_z = 0$ and a_r is independent of z and Δz. In the limit

$$a_r \to - \frac{f}{4q} \ \frac{1}{\rho} \ \frac{\partial P}{\partial r} \quad ,$$

which gives the proper limit if f/q = 4. So we conclude that for FGI-q to work properly in the cylindrical problem, it is sufficient that f/q = 4.

Summary. For the q method in the cylindrical problem all gradients give a_z = 0 and a_r is independent of z and Δz, regardless of the assumptions concerning q and f. To achieve the proper limit, it is sufficient to take f/q = 2, 2, 4 in IGT, IGA, FGI, respectively. The values of f and q required by the plane problem in Sec. VI satisfy these conditions (that is, f = 1, q = 1/2, 1/2, 1/4). The comments concerning momentum conservation between points and overlapping of masses in discussing the q method for the plane problem apply here also (Sec. VI). Our general conclusion is that while study of the cylindrical problem has added no new information, it has reinforced the conclusions drawn from the plane problem.

IX. THE CYLINDRICAL PROBLEM, MAC-0 MASS METHOD

If g = h = 1/2 as determined in Sec. VII is used, the MAC-0 masses obtained from Fig. 16 are

$$M_{1+1/2} = \rho \ \frac{\phi}{2} \ \frac{\Delta z_1 \ \Delta r_2}{4} \left(2r + \frac{\Delta r_2}{2}\right)$$

$$M_{2+1/2} = \rho \ \frac{\phi}{2} \ \frac{\Delta z_3 \ \Delta r_2}{4} \left(2r + \frac{\Delta r_2}{2}\right)$$

$$M_{3+1/2} = \rho' \frac{\phi}{2} \frac{\Delta z_3 \, \Delta r_4}{4} \left(2r - \frac{\Delta r_4}{2}\right)$$

$$M_{4+1/2} = \rho' \frac{\phi}{2} \frac{\Delta z_1 \, \Delta r_4}{4} \left(2r - \frac{\Delta r_4}{2}\right) \quad . \tag{50}$$

<u>IGT-MACO</u>. Using f = 1/2 in (44), and (50), in (7)

$$\vec{a} = \frac{\sum_\ell \vec{F}_{\ell+1/2}}{\sum_\ell M_{\ell+1/2}} = \frac{\vec{i}\,\frac{\phi}{2}\,(P' - P)\, r\, (\Delta z_1 + \Delta z_3) + \vec{k}(0)}{\frac{\phi}{8}\,(\Delta z_1 + \Delta z_3)\left[\rho\Delta r_2\left(2r + \frac{\Delta r_2}{2}\right) + \rho'\,\Delta r_4\,\left(2r - \frac{\Delta r_4}{2}\right)\right]} \quad .$$

$$a_z = 0 \quad a_r = \frac{2(P' - P)}{(\rho\Delta r_2 + \rho'\Delta r_4) + \frac{\rho\Delta r_2^2 - \rho'\Delta r_4^2}{4r}} \tag{51}$$

Hence the $a_z = 0$ and a_r is independent of z, Δz. In the limit

$$a_r \rightarrow -\frac{1}{\rho}\frac{\partial P}{\partial r} \quad .$$

<u>IGA-MACO</u>. Using f = 1/2 in (46), (50), (28) in (8)

$$\vec{a} = \frac{1}{4}\sum \frac{F}{M_{\ell+1/2}} = \vec{i}\left[\frac{P_1 - P}{\rho\Delta r_2\,\left(1 + \frac{\Delta r_2}{4r}\right)} + \frac{P' - P_1}{\rho'\Delta r_4\,\left(1 - \frac{\Delta r_4}{4r}\right)}\right] + \vec{k}(0) \quad . \tag{52}$$

Again, $a_z = 0$ and a_r is independent of z, Δz. In the limit $a_r \rightarrow -\frac{1}{\rho}\frac{\partial P}{\partial r}$.

<u>FGI-MACO</u>. Using f = 1/2 in (48), (1/2) $M_{\ell+1/2}$ from (50), in (9),

$$\vec{a} = \frac{1}{4}\sum \vec{F}_\ell/[(1/2)M_{\ell-1/2} + (1/2)M_{\ell+1/2}]$$

$$= \vec{i} \left[\frac{2(P' - P)}{\rho \Delta r_2 \left(1 + \frac{\Delta r_2}{4r}\right) + \rho' \Delta r_4 \left(1 - \frac{\Delta r_4}{4r}\right)} \right] + \vec{k}(0) \tag{53}$$

Here again, $a_z = 0$ and a_r is independent of z, Δz. In the limit

$$a_r \rightarrow - \frac{1}{\bar{\rho}} \frac{\partial P}{\partial r} \quad .$$

Summary. With the MAC-0 method for defining masses for a cylindrical problem, all three gradients give $a_z = 0$, a_r independent of z, Δz, and approach the proper limit $- \frac{1}{\rho} \frac{\partial P}{\partial r}$ as $\Delta r \rightarrow 0$. All gradients conserve momentum exchange between adjacent points of the mesh, and there is no overlapping of masses.

X. THE SPHERICAL PROBLEM - q-MASS METHOD

Let us consider a section of a spherical mesh (Fig. 11) in Fig. 17.

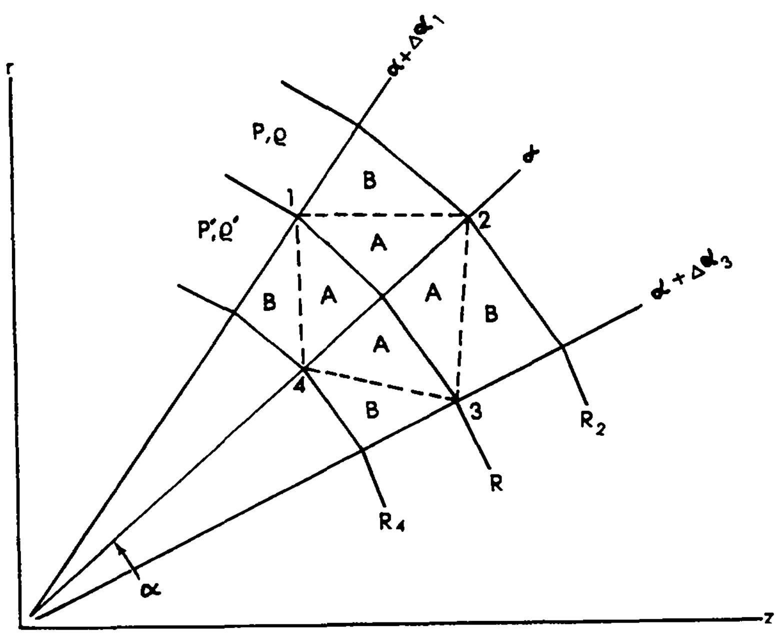

Fig. 17. A section of a spherical mesh.

Let us first calculate the masses of the zones. Each zone is divided into two triangles, A and B, as shown. Since the area formula (15) involves terms of the form

$$r_i z_j - r_j z_i$$

and since in the spherical problem all coordinates may be expressed in the form

$$r_i = R_i \sin \alpha_i \qquad r_j = R_j \sin \alpha_j$$

$$z_i = R_i \cos \alpha_i \qquad z_j = R_j \cos \alpha_j \quad ,$$

we can write

$$r_i z_j - r_j z_i = R_i R_j \left(\sin \alpha_i \cos \alpha_j - \sin \alpha_j \cos \alpha_i\right)$$

or

$$r_i z_j - r_j z_i = R_i R_j \sin \left(\alpha_i - \alpha_j\right) \quad . \tag{54}$$

Now using the area formula, (15), and (54), we can write the areas of all the triangles, all positive since $\Delta\alpha_3 < 0$,

$$A^A_{1+1/2} = \frac{1}{2} R \left(R_2 - R\right) \sin \Delta\alpha_1$$

$$A^B_{1+1/2} = \frac{1}{2} R_2 \left(R_2 - R\right) \sin \Delta\alpha_1$$

$$A^A_{2+1/2} = \frac{1}{2} R \left(R - R_2\right) \sin \Delta\alpha_3$$

$$A^B_{2+1/2} = \frac{1}{2} R_2 \left(R - R_2\right) \sin \Delta\alpha_3$$

$$A^A_{3+1/2} = \frac{1}{2} R \left(R_4 - R\right) \sin \Delta\alpha_3$$

$$A^B_{3+1/2} = \frac{1}{2} R_4 \left(R_4 - R\right) \sin \Delta\alpha_3$$

$$A^A_{4+1/2} = \frac{1}{2} R \left(R - R_4\right) \sin \Delta\alpha_1$$

$$A^B_{4+1/2} = \frac{1}{2} R_4 \left(R - R_4\right) \sin \Delta\alpha_1 \quad . \tag{55}$$

The centroids of the triangles are

$$\bar{r}^A_{1+1/2} = \frac{1}{3} \left[2R \sin \left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos \frac{\Delta\alpha_1}{2} + R_2 \sin \alpha\right]$$

$$\bar{r}^B_{1+1/2} = \frac{1}{3} \left[2R_2 \sin \left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos \frac{\Delta\alpha_1}{2} + R \sin \left(\alpha + \Delta\alpha_1\right)\right]$$

$$\bar{r}^A_{2+1/2} = \frac{1}{3} \left[2R \sin \left(\alpha + \frac{\Delta\alpha_3}{2}\right) \cos \frac{\Delta\alpha_3}{2} + R_2 \sin \alpha\right]$$

$$\bar{r}^B_{2+1/2} = \frac{1}{3} \left[2R_2 \sin \left(\alpha + \frac{\Delta\alpha_3}{2}\right) \cos \frac{\Delta\alpha_3}{2} + R \sin \left(\alpha + \Delta\alpha_3\right)\right]$$

$$\bar{r}^A_{3+1/2} = \frac{1}{3} \left[2R \sin \left(\alpha + \frac{\Delta\alpha_3}{2}\right) \cos \frac{\Delta\alpha_3}{2} + R_4 \sin \alpha\right]$$

$$\bar{r}^B_{3+1/2} = \frac{1}{3} \left[2R_4 \sin \left(\alpha + \frac{\Delta\alpha_3}{2}\right) \cos \frac{\Delta\alpha_3}{2} + R \sin \left(\alpha + \Delta\alpha_3\right)\right]$$

$$\bar{r}^A_{4+1/2} = \frac{1}{3} \left[2R \sin \left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos \frac{\Delta\alpha_1}{2} + R_4 \sin \alpha\right]$$

$$\bar{r}^B_{4+1/2} = \frac{1}{3} \left[2R_4 \sin \left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos \frac{\Delta\alpha_1}{2} + R \sin \left(\alpha + \Delta\alpha_1\right)\right] \quad . \tag{56}$$

By Theorem 5 and (55), (56)

$$m_{1+1/2} = \rho(A^A_{1+1/2}\ \bar{r}^A_{1+1/2} + A^B_{1+1/2}\ \bar{r}^B_{1+1/2})$$

$$= \frac{2}{3}\ \rho\ \frac{1}{2}\ (R_2 - R)\ \sin\ \Delta\alpha_1\ \cos\ \frac{\Delta\alpha_1}{2}\ \sin\ (\alpha + \frac{\Delta\alpha_1}{2})(R^2 + RR_2 + R_2^2)$$

$$m_{2+1/2} = \frac{2}{3}\ \rho\ \frac{1}{2}\ (R - R_2)\ \sin\ \Delta\alpha_3\ \cos\ \frac{\Delta\alpha_3}{2}\ \sin\ (\alpha + \frac{\Delta\alpha_3}{2})(R^2 + RR_2 + R_2^2)$$

$$m_{3+1/2} = \frac{2}{3}\ \rho'\ \frac{1}{2}\ (R_4 - R)\ \sin\ \Delta\alpha_3\ \cos\ \frac{\Delta\alpha_3}{2}\ \sin\ (\alpha + \frac{\Delta\alpha_3}{2})(R^2 + RR_4 + R_4^2)$$

$$m_{4+1/2} = \frac{2}{3}\ \rho'\ \frac{1}{2}\ (R - R_4)\ \sin\ \Delta\alpha_1\ \cos\ \frac{\Delta\alpha_1}{2}\ \sin\ (\alpha + \frac{\Delta\alpha_1}{2})(R^2 + RR_4 + R_4^2)\quad .$$

(57)

We then use $M_{\ell+1/2} = q\phi m_{\ell+1/2}$ as in (25) and (28), that is, assume

$$P_2 = P,\ P_4 = P' \text{ and } P_1 = P_3\quad . \tag{58}$$

<u>IGT-q</u>. From (22), using f = 1 and (58),

$$\sum_{\ell=1}^{4} \vec{F}_{\ell+1/2} = \vec{i}\ \frac{\phi}{2}\ [(P' - P)(z - z_1)(r + r_1) + 0 + (P' - P)(z_3 - z)(r_3 + r) + 0]$$

$$+ \vec{k}\ \frac{\phi}{2}\ [(P' - P)(r_1 - r)(r + r_1) + 0 + (P' - P)(r - r_3)(r + r_3) + 0]\quad .$$

(59)

From Fig. 17

$$z - z_1 = R[\cos\ \alpha - \cos\ (\alpha + \Delta\alpha_1)] = 2R\ \sin\ (\alpha + \frac{\Delta\alpha_1}{2})\ \sin\ \frac{\Delta\alpha_1}{2}$$

$$z_3 - z = -2R \sin\left(\alpha + \frac{\Delta\alpha_3}{2}\right) \sin\frac{\Delta\alpha_3}{2}$$

$$r_1 - r = 2R \cos\left(\alpha + \frac{\Delta\alpha_1}{2}\right) \sin\frac{\Delta\alpha_1}{2}$$

$$r - r_3 = -2R \cos\left(\alpha + \frac{\Delta\alpha_3}{2}\right) \sin\frac{\Delta\alpha_3}{2}$$

$$r + r_1 = 2R \sin\left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos\frac{\Delta\alpha_1}{2}$$

$$r + r_3 = 2R \sin\left(\alpha + \frac{\Delta\alpha_3}{2}\right) \cos\frac{\Delta\alpha_2}{2} \quad . \qquad (60)$$

Substituting (60) in (59), IGT-q gives

$$\vec{a} = \frac{\sum_\ell \vec{F}_{\ell+1/2}}{\sum_\ell M_{\ell+1/2}} = \frac{4R^2 \frac{\phi}{2} (P' - P)}{\sum_\ell M_{\ell+1/2}} \left\{ \vec{i} \left[\sin\frac{\Delta\alpha_1}{2} \cos\frac{\Delta\alpha_1}{2} \sin^2\left(\alpha + \frac{\Delta\alpha_1}{2}\right) \right.\right.$$

$$\left. - \sin\frac{\Delta\alpha_3}{2} \cos\frac{\Delta\alpha_3}{2} \sin^2\left(\alpha + \frac{\Delta\alpha_3}{2}\right)\right]$$

$$+ \vec{k} \left[\sin\frac{\Delta\alpha_1}{2} \cos\frac{\Delta\alpha_1}{2} \sin\left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos\left(\alpha + \frac{\Delta\alpha_1}{2}\right)\right.$$

$$\left.\left. - \sin\frac{\Delta\alpha_3}{2} \cos\frac{\Delta\alpha_3}{2} \sin\left(\alpha + \frac{\Delta\alpha_3}{2}\right) \cos\left(\alpha + \frac{\Delta\alpha_3}{2}\right)\right]\right\} \quad . \qquad (61)$$

In a spherical problem, one would expect the tangential acceleration to be zero and the radial acceleration to be constant and independent of α. Is this true of (61)?

To break (61) up into tangential and radial components, consider Fig. 18.

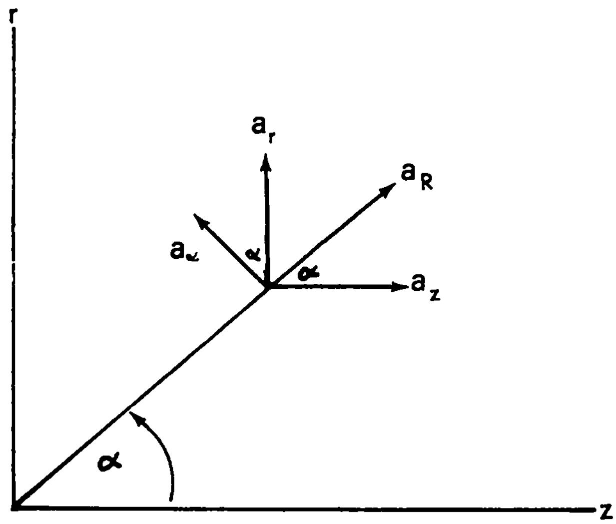

Fig. 18. Components of acceleration.

From this, we know that

$$a_\alpha = a_r \cos\alpha - a_z \sin\alpha$$

$$a_R = a_r \sin\alpha + a_z \cos\alpha \quad . \qquad (62)$$

Applying (62) to (61)

$$a_\alpha = \frac{4R^2 \frac{\phi}{2} (P' - P)}{\sum\limits_{\ell} M_{\ell+1/2}} \left\{ \left[\sin^2 \frac{\Delta\alpha_1}{2} \cos \frac{\Delta\alpha_1}{2} \sin\left(\alpha + \frac{\Delta\alpha_1}{2}\right) \right. \right.$$

$$\left. \left. - \sin^2 \frac{\Delta\alpha_3}{2} \cos \frac{\Delta\alpha_3}{2} \sin\left(\alpha + \frac{\Delta\alpha_3}{2}\right) \right] \right\}$$

$$a_R = \frac{4R^2 \frac{\phi}{2} (P' - P)}{\sum\limits_{\ell} M_{\ell+1/2}} \left\{ \left[\sin \frac{\Delta\alpha_1}{2} \cos^2 \frac{\Delta\alpha_1}{2} \sin\left(\alpha + \frac{\Delta\alpha_1}{2}\right) \right. \right.$$

$$- \sin \frac{\Delta\alpha_3}{2} \cos^2 \frac{\Delta\alpha_3}{2} \sin \left(\alpha + \frac{\Delta\alpha_3}{2}\right)\Big]\Bigg\} \quad . \tag{62a}$$

From the first relation, we see that $a_\alpha = 0$ if

$$\sin^2 \frac{\Delta\alpha_1}{2} \cos \frac{\Delta\alpha_1}{2} \sin \left(\alpha + \frac{\Delta\alpha_1}{2}\right) = \sin^2 \frac{\Delta\alpha_3}{2} \cos \frac{\Delta\alpha_3}{2} \sin \left(\alpha + \frac{\Delta\alpha_3}{2}\right) \quad . \tag{63}$$

As yet, we have found no good physical interpretation of this condition. It is well to note that it implies unequal angular spacing to achieve $a_\alpha = 0$.

Substituting for $M_{\ell+1/2}$ from (57), (58) in the expression for a_R,

$$a_R = \Big\{4R^2(P' - P) \Big[\sin \frac{\Delta\alpha_1}{2} \cos^2 \frac{\Delta\alpha_1}{2} \sin \left(\alpha + \frac{\Delta\alpha_1}{2}\right)$$

$$- \sin \frac{\Delta\alpha_3}{2} \cos^2 \frac{\Delta\alpha_3}{2} \sin \left(\alpha + \frac{\Delta\alpha_3}{2}\right)\Big]\Big\}/$$

$$\Big\{\frac{2}{3} q \left[\rho \left(R_2^3 - R^3\right) + \rho' \left(R^3 - R_4^3\right)\right] \Big[\sin \Delta\alpha_1 \cos \frac{\Delta\alpha_1}{2} \sin \left(\alpha + \frac{\Delta\alpha_1}{2}\right)$$

$$- \sin \Delta\alpha_3' \cos \frac{\Delta\alpha_3}{2} \sin \left(\alpha + \frac{\Delta\alpha_3}{2}\right)\Big]\Big\}$$

Using the trigonometric relations $\sin \Delta\alpha_1 = 2 \sin \frac{\Delta\alpha_1}{2} \cos \frac{\Delta\alpha_1}{2}$, etc., we get cancellation of all terms involving angles, so a_R becomes

$$a_R = \frac{3R^2(P' - P)}{q \left[\rho(R_2^3 - R^3) + \rho'(R^3 - R_4^3)\right]} \quad . \tag{64}$$

In the limit

$$a_R \to \frac{3R^2(P' - P)}{q2\rho\Delta R(3R^2)} = \frac{P' - P}{2q(\rho\Delta R)} \quad .$$

This approaches $-\frac{1}{\rho}\frac{\partial P}{\partial R}$ if $q = 1/2$. In conclusion, we can say that for a spherical problem, IGT-q using $f = 1$ gives a_R independent of α, $\Delta\alpha$ and approaches the proper limit for $q = 1/2$, but $a_\alpha = 0$ only if the spacing in α is defined according to (63).

IGA-q. From (20b) applied to Fig. 17, using $f = 1$ and (58).

$$\vec{F}_{1+1/2} = \vec{i}\,\frac{\phi}{2}\,[(P_1 - P)(z - z_1)(r + r_1) + 0]$$

$$+ \vec{k}\,\frac{\phi}{2}\,[(P_1 - P)(r_1 - r)(r_1 + r) + 0]$$

$$\vec{F}_{2+1/2} = \vec{i}\,\frac{\phi}{2}\,[0 + (P_3 - P)(z_3 - z)(r_3 + r)]$$

$$+ \vec{k}\,\frac{\phi}{2}\,[0 + (P_3 - P)(r - r_3)(r + r_3)]$$

$$\vec{F}_{3+1/2} = \vec{i}\,\frac{\phi}{2}\,[(P_3 - P')(z - z_3)(r + r_3) + 0]$$

$$+ \vec{k}\,\frac{\phi}{2}\,[(P_3 - P')(r_3 - r)(r_3 + r) + 0]$$

$$\vec{F}_{4+1/2} = \vec{i}\,\frac{\phi}{2}\,[0 + (P_1 - P')(z_1 - z)(r + r_1)]$$

$$+ \vec{k}\,\frac{\phi}{2}\,[0 + (P_1 - P')(r - r_1)(r + r_1)] \quad . \qquad (65)$$

Using (57), (60), and $P_3 = P_1$,

$$a = \frac{1}{4}\sum_{\ell}\left(\frac{\vec{F}}{M}\right)_{\ell+1/2} = \vec{i}\left\{\frac{3R^2}{4q}\left[\frac{P_1 - P}{\rho(R_2^3 - R^3)} + \frac{P' - P_1}{\rho'(R^3 - R_4^3)}\right]\right.$$

$$\left. x\left[\frac{\sin\left(\alpha + \frac{\Delta\alpha_1}{2}\right)}{\cos\frac{\Delta\alpha_1}{2}} + \frac{\sin\left(\alpha + \frac{\Delta\alpha_3}{2}\right)}{\cos\frac{\Delta\alpha_3}{2}}\right]\right\}$$

$$+ \vec{k}\left\{\frac{3R^2}{4q}\left[\frac{P_1 - P}{\rho(R_2^3 - R^3)} + \frac{P' - P_1}{\rho'(R^3 - R_4^3)}\right]\left[\frac{\cos\left(\alpha + \frac{\Delta\alpha_1}{2}\right)}{\cos\frac{\Delta\alpha_1}{2}} + \frac{\cos\left(\alpha + \frac{\Delta\alpha_3}{2}\right)}{\cos\frac{\Delta\alpha_3}{2}}\right]\right\}$$

Now, calculating a_α, a_R, by (62a)

$$a_\alpha = \frac{3R^2}{4q}\left[\frac{P_1 - P}{\rho(R_2^3 - R^3)} + \frac{P' - P_1}{\rho'(R^3 - R_4^3)}\right]\left(\tan\frac{\Delta\alpha_1}{2} + \tan\frac{\Delta\alpha_3}{2}\right)$$

$$a_R = \frac{3R^2}{4q}\left[\frac{P_1 - P}{\rho(R_2^3 - R^3)} + \frac{P' - P_1}{\rho'(R^3 - R_4^3)}\right](1 + 1) \quad . \qquad (66)$$

From (66), it is seen that a_R is independent of α, $\Delta\alpha$, and $a_\alpha = 0$ if the spacing in α is uniform, that is, $\Delta\alpha_3 = -\Delta\alpha_1$. In the limit

$$a_R = \frac{3R^2}{2q}\left[\frac{P' - P}{\rho\Delta R(3R^2)}\right] \rightarrow -\frac{1}{\rho}\frac{\partial P}{\partial R} \text{ if } q = \frac{1}{2} \quad .$$

In conclusion, we say that for a spherical problem, IGA-q using $f = 1$ gives a_R independent of α, $\Delta\alpha$. a_R approaches the proper limit if $q = 1/2$. a_α vanishes if the angular spacing is uniform.

<u>FGI-q</u>. If (23b) is applied to Fig. 17, using $f = 1$,

$$\vec{F}_1 = \vec{i}\ [\tfrac{\phi}{2}\ (P' - P)(z - z_1)(r + r_1)] + \vec{k}\ [\tfrac{\phi}{2}\ (P' - P)(r_1 - r)(r_1 + r)]$$

$$\vec{F}_2 = \vec{i}\ (0) + \vec{k}\ (0)$$

$$\vec{F}_3 = \vec{i}\ [\tfrac{\phi}{2}\ (P' - P)(z_3 - z)(r_3 + r)] + \vec{k}\ [\tfrac{\phi}{2}\ (P' - P)(r - r_3)(r + r_3)]$$

$$\vec{F}_4 = \vec{i}\ (0) + \vec{k}\ (0)\ . \tag{67}$$

Now, using (60), (57), and (67) in (9) ,

$$\vec{a} = \frac{1}{4} \sum_{\ell} \frac{\vec{F}_\ell}{M_{\ell-1/2} + M_{\ell+1/2}} = \frac{3R^2}{4q} \left\{ \frac{(P' - P)}{[\rho(R_2^3 - R^3) + \rho'(R^3 - R_4^3)]} \right\}$$

$$\times \left\{ \vec{i} \left[\frac{\sin\left(\alpha + \frac{\Delta\alpha_1}{2}\right)}{\cos\frac{\Delta\alpha_1}{2}} + \frac{\sin\left(\alpha + \frac{\Delta\alpha_3}{2}\right)}{\cos\frac{\Delta\alpha_3}{2}} \right] + \vec{k} \left[\frac{\cos\left(\alpha + \frac{\Delta\alpha_1}{2}\right)}{\cos\frac{\Delta\alpha_1}{2}} + \frac{\cos\left(\alpha + \frac{\Delta\alpha_3}{2}\right)}{\cos\frac{\Delta\alpha_3}{2}} \right] \right\}$$

Using (62a) to get the tangential and radial accelerations

$$a_\alpha = \frac{3R^2}{4q} \frac{(P' - P)}{[\rho(R_2^3 - R^3) + \rho'(R^3 - R_4^3)]} \left(\tan\frac{\Delta\alpha_1}{2} + \tan\frac{\Delta\alpha_3}{2} \right)$$

$$a_R = \frac{3R^2\ (P' - P)}{4q[\rho(R_2^3 - R^3) + \rho'(R^3 - R_4^3)]}\ (1 + 1)\ . \tag{68}$$

Here again, a_R is independent of α, $\Delta\alpha$, and $a_\alpha = 0$ if $\Delta\Theta_3 = -\Delta\Theta_1$. In the limit,

$$a_R \to \frac{3R^2 \left(P' - P\right)}{2q\left[2\rho\Delta R\left(3R^2\right)\right]} \to -\frac{1}{\rho}\frac{\partial P}{\partial R} \quad \text{for } q = \frac{1}{4} \quad .$$

In conclusion, we can say that for a spherical problem, FGI-q using f = 1 give a_R independent of α, $\Delta\alpha$. a_R approaches the proper limit for q = 1/4. a_α vanishes if the angular spacing is uniform.

Summary. For the q-Mass method in the spherical problem, and taking f = 1 (as required by the plane problem), IGA-q and FGI-q will give perfectly spherical motion, that is, $a_\alpha = 0$ and a_R independent of α, $\Delta\alpha$, if the spacing in α is uniform $\left(\Delta\alpha_3 = -\Delta\alpha_1\right)$. In agreement with the plane and cylindrical cases, these accelerations approach the proper limit for a_R as the spacing gets small, provided q = 1/2, 1/4 respectively. It is possible with much tedious algebra to show the reverse, that is, if the spacing in α is uniform, then it is necessary to choose f = 1 to achieve true spherical motion. This work is not shown here.

IGT-q has some differences. To achieve perfectly spherical motion with f = 1, the spacing in α must satisfy the rather complicated expression (63), for which no obvious physical interpretation has been found. It is suspected that it may be a center of mass-type condition. However, the value of a_R is uniform regardless of the spacing. The condition (63) is needed to make a_α = 0. Here again, as in the plane problem, q = 1/2 gives the proper limit.

In practice, we have found the nonspherical motion in IGT-q to be small for the types of uniform angular spacing commonly used ($\Delta\alpha \leq 6°$). The conservation of momentum discussions in the summary of Sec. VI, apply here also. As in the plane problem, the q method does not give complete conservation of momentum between adjacent vertices, and there is an overlapping of the masses used to derive the accelerations for adjacent points. The MAC-0 mass method discussed in the following section does not suffer from these disadvantages.

XI. THE SPHERICAL PROBLEM - MAC-0 MASS METHOD

Consider the four adjacent zones of a spherical mesh, as shown in Fig. 19. We will assume f = 1/2, as was done for the plane and cylinder.

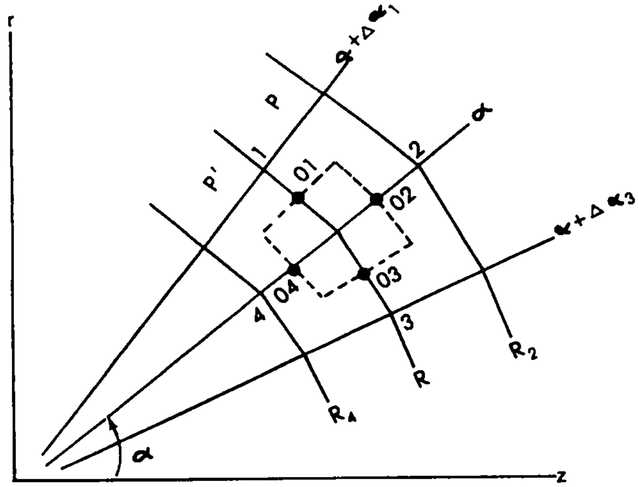

Fig. 19. The MAC-0 method in a spherical problem.

This means the midpoints of the sides will be used to partially define the surfaces and masses associated with the vertices, and hence prevent overlapping of masses while conserving momentum transfer. The coordinates of these midpoints are given by

$$r_{01} = \frac{1}{2}\left(r + r_1\right) = R \sin\left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos\frac{\Delta\alpha_1}{2}$$

$$r_{02} = \frac{1}{2}\left(R + R_2\right) \sin\alpha$$

$$r_{03} = R \sin\left(\alpha + \frac{\Delta\alpha_3}{2}\right) \cos\frac{\Delta\alpha_3}{2}$$

$$r_{04} = \frac{1}{2}\left(R + R_4\right) \sin\alpha$$

$$z_{01} = R \cos\left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos\frac{\Delta\alpha_1}{2}$$

$$z_{02} = \frac{1}{2}\left(R + R_2\right) \cos \alpha$$

$$z_{03} = R \cos \left(\alpha + \frac{\Delta\alpha_3}{2}\right) \cos \frac{\Delta\alpha_3}{2}$$

$$z_{04} = \frac{1}{2}\left(R + R_4\right) \cos \alpha \quad . \tag{69}$$

The plane problem did not completely determine how the points 8 in zones should be defined. We will look at some of the possibilities here.

<u>Average Centroids</u>. Using the definition (37) on Fig. 19, we have

$$\bar{r}_{1+1/2} = \frac{1}{4}\left\{(R + R_2)[\sin \alpha + \sin \left(\alpha + \Delta\alpha_1\right)]\right.$$

$$= \frac{1}{2}\left(R + R_2\right) \sin \left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos \frac{\Delta\alpha_1}{2} \quad ,$$

and so on, giving

$$\bar{r}_{1+1/2} = R^*_2 \sin \left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos \frac{\Delta\alpha_1}{2}$$

$$\bar{r}_{2+1/2} = R^*_2 \sin \left(\alpha + \frac{\Delta\alpha_3}{2}\right) \cos \frac{\Delta\alpha_3}{2}$$

$$\bar{r}_{3+1/2} = R^*_4 \sin \left(\alpha + \frac{\Delta\alpha_3}{2}\right) \cos \frac{\Delta\alpha_3}{2}$$

$$\bar{r}_{4+1/2} = R^*_4 \sin \left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos \frac{\Delta\alpha_1}{2}$$

$$\bar{z}_{1+1/2} = R^*_2 \cos \left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos \frac{\Delta\alpha_1}{2}$$

$$\bar{z}_{2+1/2} = R^*_2 \cos\left(\alpha + \frac{\Delta\alpha_3}{2}\right) \cos \frac{\Delta\alpha_3}{2}$$

$$\bar{z}_{3+1/2} = R^*_4 \cos\left(\alpha + \frac{\Delta\alpha_3}{2}\right) \cos \frac{\Delta\alpha_3}{2}$$

$$\bar{z}_{4+1/2} = R^*_4 \cos\left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos \frac{\Delta\alpha_1}{2} \quad ,$$

where $R^*_2 = \frac{1}{2}(R + R_2) \quad R^*_4 = \frac{1}{2}(R + R_4)$. (70)

Real Centroids. Applying (38) to Fig. 19 and omitting many steps of algebra, we get the same formulae, (70), but with

$$R^*_2 = \frac{2}{3}\frac{(R_2^3 - R^3)}{(R_2^2 - R^2)} \qquad R^*_4 = \frac{2}{3}\frac{(R^3 - R_4^3)}{(R^2 - R_4^2)} \quad . \tag{71}$$

Intersection of Diagonals. Going through the analysis for this on Fig. 19, we get the same formulae, (70), but with

$$R^*_2 = \frac{2RR_2}{R + R_2} \qquad R^*_4 = \frac{2RR_4}{R + R_4} \quad . \tag{71a}$$

Thus we see that all the various methods for defining points 8 in the zone lead to the same formula (70) for the coordinates with different definitions of R^*_2, R^*_4.

The next step is to calculate the MAC-0 masses, that is, the masses enclosed by joining the midpoints of the sides (69) to the points 8 (70). We do this by dividing each subzone into two triangles and using Theorems 4 and 5. Omitting many tedious steps, we finally arrive at

$$M_{1+1/2} = \frac{\rho\phi}{6} \cos\frac{\Delta\alpha_1}{2}\left(\sin\frac{\Delta\alpha_1}{2}\right)\{\sin\alpha\left[R^*_2\left(\frac{R + R_2}{2}\right)^2 - R^3\right]$$

$$+ \sin\left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos\frac{\Delta\alpha_1}{2} \left[R^{*2}_2 \left(\frac{R + R_2}{2}\right) - R^3\right]\}$$

$$M_{2+1/2} = \frac{\rho\phi}{6} \cos\frac{\Delta\alpha_3}{2} \left(- \sin\frac{\Delta\alpha_3}{2}\right)\{\sin\alpha \left[R^*{}_2\left(\frac{R + R_2}{2}\right)^2 - R^3\right]$$

$$+ \sin\left(\alpha + \frac{\Delta\alpha_3}{2}\right) \cos\frac{\Delta\alpha_3}{2} \left[R^{*2}_2 \left(\frac{R + R_2}{2}\right) - R^3\right]\}$$

$$M_{3+1/2} = \frac{\rho'\phi}{6} \cos\frac{\Delta\alpha_3}{2} \left(- \sin\frac{\Delta\alpha_3}{2}\right)\{\sin\alpha \left[R^3 - R^*{}_4\left(\frac{R + R_4}{2}\right)^2\right]$$

$$+ \sin\left(\alpha + \frac{\Delta\alpha_3}{2}\right) \cos\frac{\Delta\alpha_3}{2} \left[R^3 - R^{*2}_4 \left(\frac{R + R_4}{2}\right)\right]\}$$

$$M_{4+1/2} = \frac{\rho'\phi}{6} \cos\frac{\Delta\alpha_1}{2} \left(\sin\frac{\Delta\alpha_1}{2}\right)\{\sin\alpha \left[R^3 - R^*{}_4\left(\frac{R + R_4}{2}\right)^2\right]\}$$

$$+ \sin\left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos\frac{\Delta\alpha_1}{2} \left[R^3 - R^{*2}_4 \left(\frac{R + R_4}{2}\right)\right]\} \qquad (72)$$

IGT-MAC-0. Apply (22) to Fig. 19, using midpoints ($f = 1/2$)

$$\sum_{\ell=1}^{4} \vec{F}_{\ell+1/2} = \vec{i}\,\frac{\phi}{2} \left[(P' - P)(z - z_{01})(r + r_{01}) + 0\right.$$

$$\left. + (P' - P)(z_{03} - z)(r + r_{03}) + 0\right]$$

$$+ \vec{k}\,\frac{\phi}{2} \left[(P' - P)(r_{01} - r)(r + r_{01}) + 0 + (P' - P)(r - r_{03})(r + r_{03}) + 0\right] .$$

From (69), then

$$\vec{a} = \frac{\sum_{\ell} \vec{F}_{\ell+1/2}}{\sum_{\ell} M_{\ell+1/2}} = \vec{i}\left(\frac{\frac{\phi}{2}(P' - P) R^2}{\sum M_{\ell+1/2}}\right)\{[\cos\alpha - \cos\left(\alpha + \frac{\Delta\alpha_1}{2}\right)\cos\frac{\Delta\alpha_1}{2}]$$

$$x\ [\sin\alpha + \sin\left(\alpha + \frac{\Delta\alpha_1}{2}\right)\cos\frac{\Delta\alpha_1}{2}]$$

$$+ [\cos\left(\alpha + \frac{\Delta\alpha_3}{2}\right)\cos\frac{\Delta\alpha_3}{2} - \cos\alpha][\sin\alpha + \sin\left(\alpha + \frac{\Delta\alpha_3}{2}\right)\cos\frac{\Delta\alpha_3}{2}]$$

$$+ \vec{k}\left(\frac{\frac{\phi}{2}(P' - P) R^2}{\sum M_{\ell+1/2}}\right)\{[\sin\left(\alpha + \frac{\Delta\alpha_1}{2}\right)\cos\frac{\Delta\alpha_1}{2} - \sin\alpha]$$

$$x\ [\sin\alpha + \sin\left(\alpha + \frac{\Delta\alpha_1}{2}\right)\cos\frac{\Delta\alpha_1}{2}]$$

$$+ [\sin\alpha - \sin\left(\alpha + \frac{\Delta\alpha_3}{2}\right)\cos\frac{\Delta\alpha_3}{2}][\sin\alpha + \sin\left(\alpha + \frac{\Delta\alpha_3}{2}\right)\cos\frac{\Delta\alpha_3}{2}]\} \quad .$$

Now using (62)

$$a_\alpha = \frac{\frac{\phi}{2}(P' - P) R^2}{\sum M_{\ell+1/2}} \{[\sin\alpha + \sin\left(\alpha + \frac{\Delta\alpha_1}{2}\right)\cos\frac{\Delta\alpha_1}{2}]\left(\sin^2\frac{\Delta\alpha_1}{2}\right)$$

$$- [\sin\alpha + \sin\left(\alpha + \frac{\Delta\alpha_3}{2}\right)\cos\frac{\Delta\alpha_3}{2}]\left(\sin^2\frac{\Delta\alpha_3}{2}\right)\} \tag{73}$$

Thus the condition for $a_\alpha = 0$ is that the spacing in α satisfy

$$\sin^2\frac{\Delta\alpha_1}{2}[\sin\alpha + \sin\left(\alpha + \frac{\Delta\alpha_1}{2}\right)\cos\frac{\Delta\alpha_1}{2}]$$

$$= \sin^2\frac{\Delta\alpha_3}{2}[\sin\alpha + \sin\left(\alpha + \frac{\Delta\alpha_3}{2}\right)\cos\frac{\Delta\alpha_3}{2}] \quad . \tag{74}$$

This looks similar to (63) and here again I see no obvious physical interpretation. Again using (62),

$$a_R = \left(\frac{\frac{\phi}{2} (P' - P) R^2}{\sum M_{\ell+1/2}} \right)$$

$$\times \{[\sin\alpha + \sin(\alpha + \frac{\Delta\alpha_1}{2}) \cos\frac{\Delta\alpha_1}{2}] (\sin\frac{\Delta\alpha_1}{2} \cos\frac{\Delta\alpha_1}{2})$$

$$- [\sin\alpha + \sin(\alpha + \frac{\Delta\alpha_3}{2}) \cos\frac{\Delta\alpha_3}{2}] (\cos\frac{\Delta\alpha_3}{2} \sin\frac{\Delta\alpha_3}{2})\} \quad .$$

Now evaluate $\sum\limits_{\ell} M_{\ell+1/2}$ from (72). It is obvious from the last equation that we need to get terms like $[\sin\alpha + \sin(\alpha + \frac{\Delta\alpha_1}{2}) \cos\frac{\Delta\alpha_1}{2}]$, etc. A look at R* in (70), (71), (71a) shows that only the average centroid, (70), will give terms like this in $M_{\ell+1/2}$. Hence, using (70) in (72),

$$\sum_{\ell} M_{\ell+1/2} = \frac{\phi}{6} \{\rho[(\frac{R + R_2}{2})^3 - R^3] + \rho'[R^3 - (\frac{R + R_4}{2})^3]\}$$

$$\times \{\cos\frac{\Delta\alpha_1}{2} \sin\frac{\Delta\alpha_1}{2} [\sin\alpha + \sin(\alpha + \frac{\Delta\alpha_1}{2}) \cos\frac{\Delta\alpha_1}{2}]$$

$$- \cos\frac{\Delta\alpha_3}{2} \sin\frac{\Delta\alpha_3}{2} [\sin\alpha + \sin(\alpha + \frac{\Delta\alpha_3}{2}) \cos\frac{\Delta\alpha_3}{2}]\} \quad .$$

From this, we get

$$a_R = \frac{3R^2 (P' - P)}{\rho [(\frac{R + R_2}{2})^3 - R^3] + \rho' [R^3 - (\frac{R + R_4}{2})^3]} \quad , \qquad (75)$$

which is independent of α, $\Delta\alpha$, and in the limit $a_R \to -\frac{1}{\rho}\frac{\partial P}{\partial R}$. In conclusion, we can say that using the masses enclosed by joining the midpoints of the sides (f = 1/2) to the <u>average centroid</u> for the spherical problem, IGT-MAC-0, gives an a_R which is independent of α, $\Delta\alpha$ and approaches the proper limit as the spacing becomes smaller. However, $a_\alpha = 0$ only if the α spacing obeys (74).

<u>IGA-MAC-0</u>. Applying (20b) to Fig. 19, along with (58) and using the midpoints (f = 1/2),

$$\vec{F}_{1+1/2} = \vec{i}\,\frac{\phi}{2}\,[(P_1 - P)(z - z_{01})(r + r_{01}) + 0]$$

$$+ \vec{k}\,\frac{\phi}{2}\,[(P_1 - P)(r_{01} - r)(r + r_{01}) + 0]$$

$$\vec{F}_{2+1/2} = \vec{i}\,\frac{\phi}{2}\,[0 + (P_3 - P)(z_{03} - z)(r + r_{03})]$$

$$+ \vec{k}\,\frac{\phi}{2}\,[0 + (P_3 - P)(r - r_{03})(r + r_{03})]$$

$$\vec{F}_{3+1/2} = \vec{i}\,\frac{\phi}{2}\,[(P_3 - P')(z - z_{03})(r + r_{03}) + 0]$$

$$+ \vec{k}\,\frac{\phi}{2}\,(P_3 - P')(r_{03} - r)(r + r_{03}) + 0]$$

$$\vec{F}_{4+1/2} = \vec{i}\,\frac{\phi}{2}\,[0 + (P_1 - P')(z_{01} - z)(r + r_{01})]$$

$$+ \vec{k}\,\frac{\phi}{2}\,[0 + (P_1 - P')(r - r_{01})(r + r_{01})] \quad . \qquad (76)$$

Using (69), (72) and <u>average centroids</u> to define the masses for the same reasons as in IGT,

$$\vec{a} = \frac{1}{4} \sum_{\ell} \frac{\vec{F}_{\ell+1/2}}{M_{\ell+1/2}} = \vec{i}\,\frac{1}{4} \left\{ \left[\frac{3R^2(P_1 - P)}{\rho \left[\left(\frac{R + R_2}{2}\right)^3 - R^3 \right]} + \frac{3R^2(P' - P_1)}{\rho' \left[R^3 - \left(\frac{R + R_4}{2}\right)^3 \right]} \right] \right.$$

$$\left. \times \left[\frac{\cos\alpha - \cos\left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos\frac{\Delta\alpha_1}{2}}{\cos\frac{\Delta\alpha_1}{2} \sin\frac{\Delta\alpha_1}{2}} + \frac{\cos\alpha - \cos\left(\alpha + \frac{\Delta\alpha_3}{2}\right) \cos\frac{\Delta\alpha_3}{2}}{\cos\frac{\Delta\alpha_3}{2} \sin\frac{\Delta\alpha_3}{2}} \right] \right\}$$

$$+ \vec{k}\,\frac{1}{4} \left\{ \left[\frac{3R^2(P_1 - P)}{\rho \left[\left(\frac{R + R_2}{2}\right)^3 - R^3 \right]} + \frac{3R^2(P' - P_1)}{\rho' \left[R^3 - \left(\frac{R + R_4}{2}\right)^3 \right]} \right] \right.$$

$$\left. \times \left[\frac{\sin\left(\alpha + \frac{\Delta\alpha_1}{2}\right) \cos\frac{\Delta\alpha_1}{2} - \sin\alpha}{\cos\frac{\Delta\alpha_1}{2} \sin\frac{\Delta\alpha_1}{2}} + \frac{\sin\left(\alpha + \frac{\Delta\alpha_3}{2}\right) \cos\frac{\Delta\alpha_3}{2} - \sin\alpha}{\cos\frac{\Delta\alpha_3}{2} \sin\frac{\Delta\alpha_3}{2}} \right] \right\} . \quad (77)$$

Now evaluate the tangential and normal accelerations by (62)

$$a_\alpha = \left\{ \frac{3R^2(P_1 - P)}{4\rho \left[\left(\frac{R + R_2}{2}\right)^3 - R^3 \right]} + \frac{3R^2(P' - P_1)}{4\rho' \left[R^3 - \left(\frac{R + R_4}{2}\right)^3 \right]} \right\} \left(\tan\frac{\Delta\alpha_1}{2} + \tan\frac{\Delta\alpha_3}{2} \right)$$

$$a_R = \left\{ \frac{3R^2(P_1 - P)}{4\rho \left[\left(\frac{R + R_2}{2}\right)^3 - R^3 \right]} + \frac{3R^2(P' - P_1)}{4\rho' \left[R^3 - \left(\frac{R + R_4}{2}\right)^3 \right]} \right\} (1 + 1) \quad . \quad (78)$$

From these relations, we note that a_R is independent of α, $\Delta\alpha$, and in the limit

$$a_R \rightarrow \frac{(3R^2)\,(P' - P)}{\rho(3R^2)\,\Delta R} \rightarrow -\frac{1}{\bar{\rho}} \frac{\partial P}{\partial R} \quad ,$$

but $a_\alpha = 0$ only if the spacing in α is uniform.

In conclusion, we can say that using the masses enclosed by joining the midpoints of the sides ($f = 1/2$) to the average centroid for the spherical problem, IGA-MACO, gives an a_R which is independent of α, $\Delta\alpha$ and approaches the proper limit for small spacing. However, $a_\alpha = 0$ only if the spacing in α is uniform.

FGI-MAC-0. From (23b) applied to Fig. 19, using the midpoints ($f = 1/2$)

$$\vec{F}_1 = \vec{i}\,\frac{\phi}{2}\left[(P' - P)(z - z_{01})(r + r_{01})\right] + \vec{k}\,\frac{\phi}{2}\left[(P' - P)(r_{01} - r)(r_{01} + r)\right]$$

$$\vec{F}_2 = 0$$

$$\vec{F}_3 = \vec{i}\,\frac{\phi}{2}\left[(P' - P)(z_{03} - z)(r + r_{03})\right] + \vec{k}\,\frac{\phi}{2}\left[(P' - P)(r - r_{03})(r + r_{03})\right]$$

$$\vec{F}_4 = 0 \quad . \tag{79}$$

Here again we use (69) and (72) with the average centroids and $\frac{1}{2} M_{\ell+1/2}$ as required for FGI (Fig. 15), giving

$$\vec{a} = \frac{1}{4}\sum_{\ell} \frac{\vec{F}_\ell}{\frac{1}{2}\left(M_{\ell-1/2} + M_{\ell+1/2}\right)} = \frac{\frac{3}{2} R^2 \quad (P' - P)}{\rho\left[\left(\frac{R + R_2}{2}\right)^3 - R^3\right] + \rho'\left[R^3 - \left(\frac{R + R_4}{2}\right)^3\right]}$$

$$\times\left\{\vec{i}\left[\frac{\cos\alpha - \cos\left(\alpha + \frac{\Delta\alpha_1}{2}\right)\cos\frac{\Delta\alpha_1}{2}}{\cos\frac{\Delta\alpha_1}{2}\sin\frac{\Delta a_1}{2}} + \frac{\cos\alpha - \cos\left(\alpha + \frac{\Delta\alpha_3}{2}\right)\cos\frac{\Delta\alpha_3}{2}}{\cos\frac{\Delta\alpha_3}{2}\sin\frac{\Delta\alpha_3}{2}}\right]\right.$$

$$\left. + \vec{k}\left[\frac{\sin\left(\alpha + \frac{\Delta\alpha_1}{2}\right)\cos\frac{\Delta\alpha_1}{2} - \sin\alpha}{\cos\frac{\Delta\alpha_1}{2}\sin\frac{\Delta\alpha_1}{2}} + \frac{\sin\left(\alpha + \frac{\Delta\alpha_3}{2}\right)\cos\frac{\Delta\alpha_3}{2} - \sin\alpha}{\cos\frac{\Delta\alpha_3}{2}\sin\frac{\Delta\alpha_3}{2}}\right]\right\} . \tag{80}$$

Except for the coefficient, this is the same as (77), so

$$a_\alpha = \frac{\frac{3}{2} R^2 (P' - P) \left(\tan \frac{\Delta\alpha_1}{2} + \tan \frac{\Delta\alpha_3}{2}\right)}{\left\{\rho \left[\left(\frac{R + R_2}{2}\right)^3 - R^3\right] + \rho' \left[R^3 - \left(\frac{R + R_4}{2}\right)^3\right]\right\}}$$

$$a_R = \frac{\frac{3}{2} R^2 (P' - P)(1 + 1)}{\left\{\rho \left[\left(\frac{R + R_2}{2}\right)^3 - R^3\right] + \rho' \left[R^3 - \left(\frac{R + R_4}{2}\right)^3\right]\right\}} . \qquad (81)$$

Here again, a_R is independent of α, $\Delta\alpha$, and in the limit

$$a_R \rightarrow \frac{3R^2 \;(P' - P)}{\rho \;3R^2 \;\Delta R} \rightarrow - \frac{1}{\rho} \frac{\partial P}{\partial R}$$

and $a_\alpha = 0$ if the spacing in α is uniform.

In conclusion, we can say that use of half the MAC-0 masses in the spherical problem for FGI-MAC-0 gives an a_R which is independent of α, $\Delta\alpha$ and approaches the proper limit for small spacing. However, $a_\alpha = 0$ only if the angular spacing is equal.

Summary. In the spherical problem, it appears best to define the MAC-0 masses as those masses enclosed by straight lines joining the midpoints ($f = 1/2$) of the sides to the average centroid defined by (37). (In fact the abbreviation MAC-0 has been derived as an abbreviated description of Midpoint, Average Centroid.) Under this definition, all these integrated gradients, IGT, IGA, FGI, give a radial acceleration, a_R, for the spherical problem which is independent of angle α and angular spacing $\Delta\alpha$, and all approach the proper limit $- \frac{1}{\rho} \frac{\partial P}{\partial R}$. To get $a_\alpha = 0$, it is necessary to use constant α spacing in IGA and FGI, and the condition (74) for IGT. In practice, we have found that uniform angular spacing in IGT (for $\Delta\alpha \leq 6°$) gives very small nonspherical motions.

The conservation of momentum discussions for the plane problem apply here again. The appeal of the MAC-0 method of defining masses is again evident in that it conserves momentum transfer exactly between each vertex and its eight neighbors, and does not allow any overlapping of the masses associated with adjacent vertices.

XII. THE BOUNDARY CASES

In any numerical calculation there are boundaries, and it is necessary to treat these boundaries properly, for any perturbation in these points frequently propagates throughout the mesh, often with magnification of the error. Contrary to what one might expect, setting up proper calculations at a boundary is not necessarily simpler than the general case and is often more difficult, as many people with computing experience will testify. In fact, it was inability to understand the boundary cases which prompted this whole study of gradients for two-dimensional Lagrangian hydrodynamics.

In general, by a boundary we will mean any point for which the general gradients are not applicable unless special assumptions are made. Examples could be a free surface, points constrained to move along a particular line or surface, etc. In this section, we will confine ourselves to the latter, leaving the free surface case to be discussed in the following section.

Suppose we consider a point 0 which is constrained to move along a line (representing a surface in the cylindrical system) in the r, z plane (Fig. 20). The most common examples of this are motion along a line, r = constant (including the z axis), and motion along a line, z = constant (including the r axis). There are a number of ways this point could be treated, of which three will be mentioned.

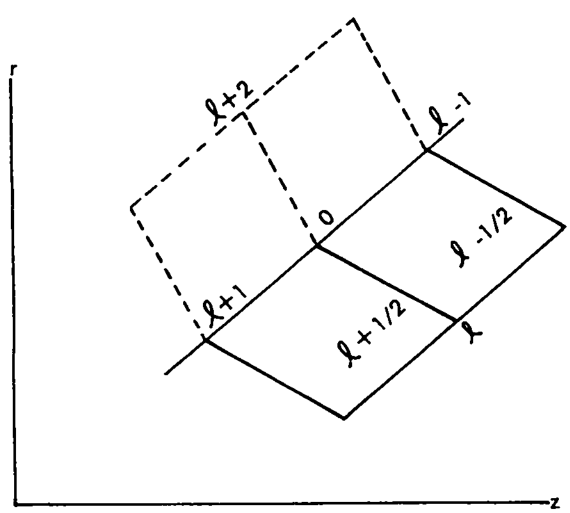

Fig. 20. A boundary point.

B1 The point 0 can be moved along the line $(\ell+1) - 0 - (\ell-1)$ with the same velocity that point ℓ has in that direction. We will say no more about this method, except to remark that since it prevents the mesh at the boundary from becoming badly distorted, it can be used as a last resort if other methods fail.

B2 The zones $\ell + 1/2$, $\ell - 1/2$ can be reflected across the line (dashed lines in Fig. 20) to give fictitious $\ell + \frac{3}{2}$, $\ell - \frac{3}{2}$, which have the same pressures and masses as their real counterparts. However, care must be taken to use $(r + r_{\ell+2}) = (r + r_{\ell})$ in calculating the forces on the surface $0 - (\ell+2)$ in fictitious zones so that the area over which the pressures act will be the same as the area of surface $0 - \ell$. Care must also be taken with the masses. This can be a bit confusing. This method has the advantage that it makes the acceleration normal to the line vanish, as desired.

B3 The gradients can be calculated using only the zones involved. If we use zones $\ell + 1/2$, $\ell +1/2$, taking $P_{\ell-1} = P_{\ell-1/2}$, $P_{\ell+1} = P_{\ell+1/2}$ in (20b), and summing and averaging over only the two zones involved, we get an acceleration which is not necessarily in the desired direction. However, we can keep only the component which is tangential to the line $(\ell+1) - 0 - (\ell-1)$ and set the normal component equal to zero. This method is exactly equivalent to B2, because for

IGT, B2 gives twice the force divided by twice the mass; for IGA, FGI, B2 gives twice the force divided by 4 rather than 2.

We will use B3 because it involves fewer terms and avoids the difficulties of meddling with the $(r + r_{\ell+2})$ of the reflected zones. The following sections will contain discussions of some of the boundary cases in the plane, cylinder, and sphere. We will show that similar assumptions for f, q, etc., are required to give the same accelerations for boundary points as for the general points. This reinforces the arguments advanced in the discussion of the general points and leads to a consistent model, except for FGI, where some of the assumptions to be made are not clearly evident.

XIIA. MOVEMENT CONSTRAINED TO A LINE r = CONSTANT

Consider the case (Fig. 21) where point 0 is to slide along the line r = constant.

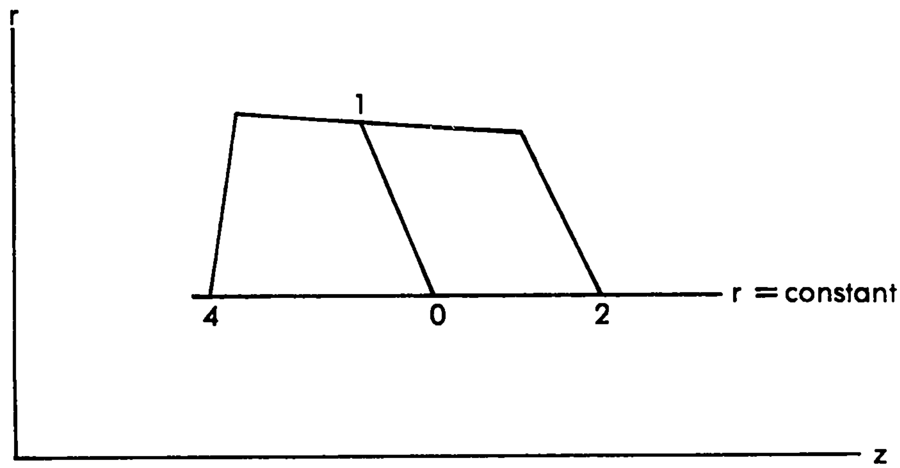

Fig. 21. Sliding boundary.

To use B3 we assume $a_r = 0$ and $P_4 = P_{4+1/2}$, $P_2 = P_{1+1/2}$. Then from (20b)

<u>IGT.</u>

$$\vec{a} = a_z = \left(\frac{F_{4+1/2} + F_{1+1/2}}{M_{4+1/2} + M_{1+1/2}} \right)$$

$$a_z = \frac{\phi}{2} \left(P_{4+1/2} - P_{1+1/2}\right)(r_1 - r)(r_1 + r)/\left(M_{4+1/2} + M_{1+1/2}\right) \quad . \qquad (82)$$

<u>IGA.</u>

$$\vec{a} = a_z = \frac{1}{2} \left[\left(\frac{F}{M}\right)_{4+1/2} + \left(\frac{F}{M}\right)_{1+1/2} \right]$$

$$a_z = \frac{1}{2} \frac{\phi}{2} \left[\left(P_1 - P_{4+1/2}\right)(r - r_1)(r + r_1)/M_{4+1/2} \right.$$

$$\left. + \left(P_1 - P_{1+1/2}\right)(r_1 - r)(r_1 + r)/M_{1+1/2} \right] \quad . \qquad (83)$$

<u>FGI.</u>

$$\vec{a} = a_z = \frac{1}{2} \frac{F_1}{M_{4+1/2} + M_{1+1/2}}$$

$$a_z = \frac{1}{2}\frac{\phi}{2}\,[(P_{4+1/2} - P_{1+1/2})(r_1 - r)(r_1 + r)]/(M_{4+1/2} + M_{1+1/2}). \quad (84)$$

The use of 1/L = 1/2 is not obvious for FGI, but it is necessary to achieve the correct answer in later steps.

We now apply these formulae to the plane problem for the q-mass method, using masses from (26) and notations of Fig. 12.

IGT-q: (82) becomes

$$a_z = \frac{\frac{\phi}{2}(P' - P)(f\Delta r_1)(2r + f\Delta r_1)}{q\,\frac{\phi}{2}\,\Delta r_1(2r + \Delta r_1)(\rho'\Delta z_4 + \rho\Delta z_2)} \quad .$$

This agrees with the general case (30) if f = 1, as required also in (30).

IGA-q: (83) becomes

$$a_z = \frac{1}{2}\frac{\phi}{2}\left[\frac{(P_1 - P')(-f\Delta r_1)(2r + f\Delta r_1)}{q\,\frac{\phi}{2}\,\rho'\Delta r_1\,\Delta z_4(2r + \Delta r_1)} + \frac{(P_1 - P)\,(f\Delta r_1)(2r + f\Delta r_1)}{q\,\frac{\phi}{2}\,\rho\Delta r_1\Delta z_2(2r + \Delta r_1)}\right],$$

which agrees with the general case (32) if f = 1, as required also in (32).

FGI-q: (84) becomes

$$a_z = \frac{1}{2}\frac{\phi}{2}\left[\frac{(P' - P)(f\Delta r_1)(2r + f\Delta r_1)}{q\,\frac{\phi}{2}\,\Delta r_1(2r + \Delta r_1)(\rho'\Delta z_4 + \rho\Delta z_2)}\right] \quad ,$$

which agrees with (35) for f = 1, as required there also.

Applying (82), (83, (84) to the plane problem for the MAC-0 mass method, using masses from (39),

IGT-MAC-0. (82) becomes

$$a_z = \frac{\frac{\phi}{2}\,(P' - P)\,(f\Delta r_1)\,(2r + f\Delta r_1)}{\frac{\phi}{2}\,(\frac{\Delta r_1}{2})\,(2r + \frac{\Delta r_1}{2})\,\frac{1}{2}\,(\rho'\Delta z_4 + \rho\Delta z_2)} \quad ,$$

which agrees with (40) for f = 1/2.

IGA-MAC-0. (83) becomes

$$a_z = \frac{1}{2}\frac{\phi}{2}\left[\frac{(P' - P_1)\,(f\Delta r_1)\,(2r + f\Delta r_1)}{\frac{\phi}{2}\,(f\Delta r_1)\,(\frac{\rho'\Delta z_4}{2})\,(2r + f\Delta r_1)} + \frac{(P_1 - P)\,(f\Delta r_1)\,(2r + f\Delta r_1)}{\frac{\phi}{2}\,(f\Delta r_1)\,(\frac{\rho\Delta z_2}{2})\,(2r + f\Delta r_1)}\right] .$$

which agrees with (41) for f = 1/2.

FGI-MAC-0. (84) becomes, using half of M's from (39) (see Fig. 15c),

$$a_z = \frac{1}{2}\frac{\phi}{2}\left[\frac{(P' - P)\,(f\Delta r_1)\,(2r + f\Delta r_1)}{\frac{1}{2}\frac{\phi}{2}\,(f\Delta r_1)\,(2r + f\Delta r_1)\,\frac{1}{2}\,(\rho'\,\Delta z_4 + \rho\,\Delta z_2)}\right] ,$$

which agrees with (42) for f = 1/2.

The general conclusion for the plane problem is that boundary cases derived by B3 for motion along a line r = constant give exact agreement of accelerations with the general case under the same assumptions for f, q, etc., for both the q method and the MAC-0 method.

Let us now apply (82), (83), (84) to the spherical problem for motion of the boundary points along the z axis. We will use notation of Fig. 17 with $\alpha = 0$ and zones 2+1/2, 3+1/2 omitted. Using the B3 boundary method, $P_4 = P_{4+1/2}$, $P_2 = P_{1+1/2}$, and $a_r = 0$. Equations (82), (83), (84) are still applicable except now r = 0.

For the q-mass method, with f = 1, we get from (60) and (57)

$$r_1 - r = r + r_1 = 2R \cos\frac{\Delta\alpha_1}{2} \sin\frac{\Delta\alpha_1}{2}$$

$$m_{1+1/2} = \frac{2}{3}\,\rho\,\frac{1}{2}\,(R_2^3 - R^3)\;2 \sin^2\frac{\Delta\alpha_1}{2} \cos^2\frac{\Delta\alpha_1}{2}$$

$$m_{4+1/2} = \frac{2}{3}\,\rho'\,\frac{1}{2}\,(R^3 - R_4^3)\;2 \sin^2\frac{\Delta\alpha_1}{2} \cos^2\frac{\Delta\alpha_1}{2} .$$

Using these, we get

IGT-q: From (82)

$$a_R = a_z = \frac{\frac{\phi}{2}(P' - P)\, 4R^2 \cos^2\frac{\Delta\alpha_1}{2}\sin^2\frac{\Delta\alpha_1}{2}}{q\,\frac{\phi}{2}\,\frac{2}{3}\, 2\sin^2\frac{\Delta\alpha_1}{2}\cos^2\frac{\Delta\alpha_1}{2}\left[\rho'(R^3 - R_4^3) + \rho(R_2^3 - R^3)\right]},$$

which is identical to a_R for the spherical case given by (64).

IGA-q. From (83)

$$a_R = a_z = \frac{\frac{1}{2}\,\frac{\phi}{2}(P' - P_1)\, 4R^2 \sin^2\frac{\Delta\alpha_1}{2}\cos^2\frac{\Delta\alpha_1}{2}}{q\,\frac{\phi}{2}\,\frac{2}{3}\,\rho'(R^3 - R_4^3)\, 2\sin^2\frac{\Delta\alpha_1}{2}\cos^2\frac{\Delta\alpha_1}{2}}$$

$$+ \frac{\frac{1}{2}\,\frac{\phi}{2}(P_1 - P)\quad 4R^2 \sin^2\frac{\Delta\alpha_1}{2}\cos^2\frac{\Delta\alpha_1}{2}}{q\,\frac{\phi}{2}\,\frac{2}{3}\,\rho\,(R_2^3 - R^3)\, 2\sin^2\frac{\Delta\alpha_1}{2}\cos^2\frac{\Delta\alpha_1}{2}},$$

which is identical to (66).

FGI-q. From (84)

$$a_R = a_z = \frac{\frac{1}{2}\,\frac{\phi}{2}(P' - P)\, 4R^2 \sin^2\frac{\Delta\alpha_1}{2}\cos^2\frac{\Delta\alpha_1}{2}}{q\,\frac{\phi}{2}\,\frac{2}{3}\, 2\sin^2\frac{\Delta\alpha_1}{2}\cos^2\frac{\Delta\alpha_1}{2}\left[\rho'(R^3 - R_4^3) + \rho(R_2^3 - R^3)\right]},$$

which agrees with (68).

For the MAC-0 method, for the spherical problem along the z axis, we get from (69) and (72), taking f = 1/2,

$$r_{01} - r = r_{01} + r = R\sin\frac{\Delta\alpha_1}{2}\cos\frac{\Delta\alpha_1}{2}$$

$$M_{4+1/2} = \rho'\,\frac{\phi}{6}\cos^2\frac{\Delta\alpha_1}{2}\sin^2\frac{\Delta\alpha_1}{2}\left[R^3 - R_4^{*2}\left(\frac{R + R_4}{2}\right)\right]$$

$$M_{1+1/2} = \rho \frac{\phi}{6} \cos^2 \frac{\Delta\alpha_1}{2} \sin^2 \frac{\Delta\alpha_1}{2} [R_2^{*2} (\frac{R + R_2}{2}) - R^3] \quad .$$

Using these with R_2^*, R_4^* defined as required for the general cases,

IGT-MAC-0. (82) gives

$$a_R = a_z = \frac{\frac{\phi}{2} (P' - P) R^2 \sin^2 \frac{\Delta\alpha_1}{2} \cos^2 \frac{\Delta\alpha_1}{2}}{\frac{\phi}{6} \sin^2 \frac{\Delta\alpha_1}{2} \cos^2 \frac{\Delta\alpha_1}{2} \{\rho'[R^3 - (\frac{R + R_4}{2})^3] + \rho[(\frac{R + R_2}{2})^3 - R^3]\}} ,$$

which agrees with the general case (75).

IGA-MAC-0. (83) gives

$$a_R = a_z = \frac{\frac{1}{2} \frac{\phi}{2} (P' - P_1) R^2 \sin^2 \frac{\Delta\alpha_1}{2} \cos^2 \frac{\Delta\alpha_1}{2}}{\rho' \frac{\phi}{6} \sin^2 \frac{\Delta\alpha_1}{2} \cos^2 \frac{\Delta\alpha_1}{2} [R^3 - (\frac{R + R_4}{2})^3]}$$

$$+ \frac{\frac{1}{2} \frac{\phi}{2} (P_1 - P) R^2 \sin^2 \frac{\Delta\alpha_1}{2} \cos^2 \frac{\Delta\alpha_1}{2}}{\rho \frac{\phi}{6} \sin^2 \frac{\Delta\alpha_1}{2} \cos^2 \frac{\Delta\alpha_1}{2} [(\frac{R + R_2}{2})^3 - R^3]} ,$$

which agrees with (78)

FGI-MAC-0. (84) gives [remembering to use (1/2)M]

$$a_R = a_z = \frac{\frac{1}{2} \frac{\phi}{2} (P' - P) R^2 \sin^2 \frac{\Delta\alpha_1}{2} \cos^2 \frac{\Delta\alpha_1}{2}}{\frac{1}{2} \frac{\phi}{6} \sin^2 \frac{\Delta\alpha_1}{2} \cos^2 \frac{\Delta\alpha_1}{2} \{\rho'[R^3 - (\frac{R + R_4}{2})^3] + \rho[(\frac{R + R_2}{2})^3 - R^3]\}} ,$$

which agrees with (81).

The general conclusion of the spherical problem is that boundary cases derived by B3 for motion along the z axis (r = constant) give exact agreement of accelerations with the general case under the same assumptions for f, q, etc., for both the q and the MAC-0 mass methods.

For motion along a line z = constant (for example, the r axis), methods very similar to those above can be used to show that boundary method B3 gives accelerations which agree exactly with the general case in the cylindrical and spherical problems for the same values of f, q, etc.

XIII. FREE SURFACES

By a free surface, we mean any point 0 which lies along a boundary $(\ell-1) - 0 - (\ell+1)$ beyond which there is no material (Fig. 22). Here again, for deriving the formulae, we could use the reflection type method,

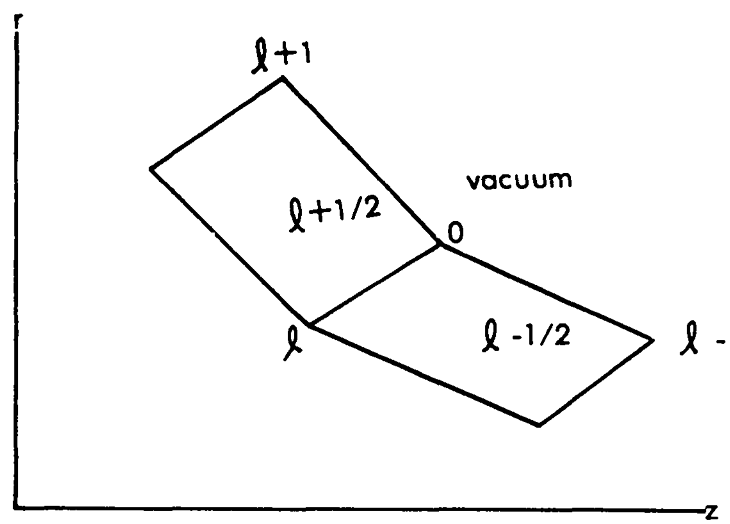

Fig. 22. Free surface.

B2, in which the fictitious reflected zones are assumed to have zero pressures and zero masses, or we can use a B3 method, where we use only the zones that actually exist. The latter gives correct formulae for IGT and IGA but requires the inclusion of an extra factor of 1/2 for FGI. The B2 method gives correct results provided one uses the definition that the indeterminate 0/0 = 0. We will use method B3.

In a Lagrangian mesh, where mass is confined to initially chosen zones, there will be no mass flow and hence no momentum flow across a free surface. Since $\int P d\vec{S}$ represents a flow of momentum, the logical assumption would appear to be that <u>on a free surface P = 0</u>. Thus in Fig. 22, at the free surface we assume

$$P_{\ell+1} = 0 \qquad P_{\ell-1} = 0 \quad . \tag{85}$$

Using the general formula (20B) on Fig. 22, taking (85) into account, we get the formulae for a free surface:

IGT.

$$\vec{a} = \frac{\vec{F}_{\ell-1/2} + \vec{F}_{\ell+1/2}}{M_{\ell-1/2} + M_{\ell+1/2}} = \vec{i}\,\frac{\phi}{2}\,[(0 - P_{\ell-1/2})(z - z_{\ell-1})(r + r_{\ell-1})$$

$$+ (P_{\ell+1/2} - P_{\ell-1/2})(z_{\ell} - z)(r_{\ell} + r)$$

$$+ (0 - P_{\ell+1/2})(z_{\ell+1} - z)(r_{\ell+1} + r)]$$

$$+ \vec{k}\,\frac{\phi}{2}\,[(0 - P_{\ell-1/2})(r_{\ell-1} - r)(r_{\ell-1} + r)$$

$$+ (P_{\ell+1/2} - P_{\ell-1/2})(r - r_{\ell})(r + r_{\ell})$$

$$+ (0 - P_{\ell+1/2})(r - r_{\ell+1})(r + r_{\ell+1})]/(M_{\ell-1/2} + M_{\ell+1/2}) \quad . \qquad (86)$$

IGA. $\vec{a} = \frac{1}{2}\left[\left(\frac{\vec{F}}{M}\right)_{\ell-1/2} + \left(\frac{\vec{F}}{M}\right)_{\ell+1/2}\right]$

$$= \vec{i}\left\{\frac{\frac{\phi}{2}}{2M_{\ell-1/2}}\,[(0 - P_{\ell-1/2})(z - z_{\ell-1})(r + r_{\ell-1})\right.$$

$$+ (P_{\ell} - P_{\ell-1/2})(z_{\ell} - z)(r_{\ell} + r)]$$

$$+ \frac{\frac{\phi}{2}}{2M_{\ell+1/2}}\,[(P_{\ell} - P_{\ell+1/2})(z - z_{\ell})(r + r_{\ell})$$

$$\left. + (0 - P_{\ell+1/2})(z_{\ell+1} - z)(r_{\ell+1} + r)]\right\}$$

$$+ \vec{k} \left\{ \frac{\frac{\phi}{2}}{2M_{\ell-1/2}} [(0 - P_{\ell-1/2})(r_{\ell-1} - r)(r_{\ell-1} + r) \right.$$

$$+ (P_{\ell} - P_{\ell-1/2})(r - r_{\ell})(r + r_{\ell})]$$

$$+ \frac{\frac{\phi}{2}}{2M_{\ell+1/2}} [(P_{\ell} - P_{\ell+1/2})(r_{\ell} - r)(r_{\ell} + r)$$

$$\left. + (0 - P_{\ell+1/2})(r - r_{\ell+1})(r + r_{\ell+1})] \right\} . \qquad (87)$$

FGI. We insert an extra factor of 1/2, and use (20b) with assumption (23a)

$$\vec{a} = \frac{1}{4} \left(\frac{\vec{F}_{\ell-1/2}}{M_{\ell-1/2}} + \frac{\vec{F}_{\ell-1/2} + \vec{F}_{\ell+1/2}}{M_{\ell-1/2} + M_{\ell+1/2}} + \frac{\vec{F}_{\ell+1/2}}{M_{\ell+1/2}} \right)$$

$$= \vec{i} \left\{ \frac{\frac{\phi}{2}}{4M_{\ell-1/2}} [(0 - P_{\ell-1/2})(z - z_{\ell-1})(r + r_{\ell-1})] \right.$$

$$+ \frac{\frac{\phi}{2}}{4(M_{\ell-1/2} + M_{\ell+1/2})} [(P_{\ell+1/2} - P_{\ell-1/2})(z_{\ell} - z)(r_{\ell} + r)]$$

$$\left. + \frac{\frac{\phi}{2}}{4M_{\ell+1/2}} [(0 - P_{\ell+1/2})(z_{\ell+1} - z)(r_{\ell+1} + r)] \right\}$$

$$+ \left\{ \vec{k} \frac{\frac{\phi}{2}}{4M_{\ell-1/2}} [(0 - P_{\ell-1/2})(r_{\ell-1} - r)(r + r_{\ell-1})] \right.$$

$$+ \frac{\frac{\phi}{2}}{4(M_{\ell-1/2} + M_{\ell+1/2})} [(P_{\ell+1/2} - P_{\ell-1/2})(r - r_{\ell})(r + r_{\ell})]$$

$$+\ \frac{\frac{\phi}{2}}{4M_{\ell+1/2}}\left[(0 - P_{\ell+1/2})(r - r_{\ell+1})(r + r_{\ell+1})\right]\Bigg\} \ . \qquad (88)$$

<u>Application to the Plane Problem.</u>

<u>q-Mass Method</u>. (Refer to Sec. VI.) We use the notation of Fig 12, taking zones 3+1/2, 4+1/2 as the real zones and using masses from (26).

<u>IGT-q</u>. Using (86) with $\ell-1 = 3$, $\ell = 4$, $\ell+1 = 1$

$$\vec{a} = \vec{i}(0) + \vec{k}\,\frac{\frac{\phi}{2}\, f(P' - 0)[\Delta r_3(2r - f\Delta r_3) + \Delta r_1(2r + f\Delta r_1)]}{\frac{\phi}{2}\,\rho'\ \Delta z_4[\Delta r_3(2r - \Delta r_3) + \Delta r_1(2r + \Delta r_1)]} \ .$$

We see that, as in the general case, (30), a_z is independent of r, Δr if f = 1, in which case

$$a_z = \frac{1}{q}\,\frac{(P' - 0)}{\rho'\Delta z_4} \to -\frac{1}{\rho}\,\frac{\partial P}{\partial z}$$

if q = 1/2. This approaches the proper limit for a free surface as compared to (30) for a general point.

<u>IGA-q</u>. Using (87) with $P_4 = P'$

$$\vec{a} = \vec{i}(0) + \vec{k}\left[\frac{\frac{\phi}{2}\,(P' - 0)\ f\Delta r_3(2r - f\Delta r_3)}{2q\,\frac{\phi}{2}\,\rho'\ \Delta r_3\Delta z_4(2r - \Delta r_3)} + \frac{\frac{\phi}{2}\,(P' - 0)\ f\Delta r_1(2r + f\Delta r_1)}{2q\,\frac{\phi}{2}\,\rho'\ \Delta r_1\Delta z_4(2r + \Delta r_1)}\right] .$$

Thus a_z is independent of r, Δr if f = 1, in which case, for q = 1/2

$$a_z = \frac{(P' - 0)(1 + 1)}{2q\rho'\Delta z_4} = \frac{(P' - 0)}{\rho'\ \frac{\Delta z_4}{2}}$$

approaches the proper limit for a free surface.

<u>FGI-q</u>. Using (88) if f = 1, q = 1/4 as in the general case

$$\vec{a} = \vec{i}(0) + \vec{k}\left[\frac{\frac{\phi}{2}\left(P' - 0\right) f\Delta r_3\left(2r - f\Delta r_3\right)}{4q\,\frac{\phi}{2}\,\rho'\,\Delta r_3\Delta z_4\left(2r - \Delta r_3\right)} + \frac{\frac{\phi}{2}\left(P' - 0\right) f\Delta r_1\left(2r + f\Delta r_1\right)}{4q\,\frac{\phi}{2}\,\rho'\,\Delta r_1\Delta z_4\left(2r + \Delta r_1\right)}\right].$$

whence a_z is independent of r, Δr if $f = 1$ so that

$$\vec{a} = \vec{i}(0) + \vec{k}\left[\frac{(P' - 0)(1 + 1)}{4q\,\rho'\Delta z_4}\right] .$$

If $q = \frac{1}{4}$, $a_z = \dfrac{P' - 0}{\rho'\,\frac{\Delta z_4}{2}}$ gives the proper limit for a free surface.

MAC-0 MASS METHOD. (Refer to Sec. VII.) Again using Fig. 12, but masses from (39), omitting the details, we get agreement with the general case. For example, to have a_z independent of r, Δr, it is necessary to take $f = 1/2$. The results are

IGT-MAC-0. (86), with $f = 1/2$, becomes

$$\vec{a} = a_z = \frac{P' - 0}{\rho'\,\frac{\Delta z_4}{2}},$$ which approaches the proper limit.

IGA-MAC-0. (87), with $f = 1/2$, becomes

$$\vec{a} = a_z = \frac{1}{2}\;\frac{(P' - 0)(1 + 1)}{\rho'\,\frac{\Delta z_4}{4}} = \frac{(P' - 0)}{\rho'\,\frac{\Delta z_4}{2}} .$$

FGI-MAC-0. (88), with $f = 1/2$, and as customary using $(1/2)M$ from (39)

$$\vec{a} = a_z = \frac{1}{4}\;\frac{(P' - 0)(1 + 1)}{\rho'\,\frac{\Delta z_4}{4}} = \frac{(P' - 0)}{\rho'\,\frac{\Delta z_4}{2}} .$$

Cylindrical Problem. Similar derivations can be carried through for the cylindrical problem, using Fig. 16 and masses from (43) and (50). Taking

$\ell = 4$ in Fig. 16 and applying (86), (87), and (88), one gets $a_z = 0$ and an expression for a_r in terms of f for all gradients. As in the general case, it is found that for the q-mass method, it is necessary to take f = 1 and q = 1/2, 1/2, 1/4 for the three gradients to get the proper limiting case. For the MAC-0 mass method, it is necessary to take f = 1/2 in all gradients and again to use (1/2)M in FGI.

Spherical Problem. Referring to Secs. X and XI, Figs. 17 and 19, we use only points ℓ = 3, 4, 1 and assume the two outer zones are not present. Again we use $P_{3+1/2} = P_{4+1/2} = P'$, so (86), (87), and (88) simplify a great deal.

q-Mass Method. Use f = 1 and masses from (57).

IGT-q. With these assumptions (86) gives the same result as (59) with P = 0, so the conclusions of the general case apply here. Hence, $a_\alpha = 0$ only if (63) holds, and a_R is given by (64) with P = 0 and $M_{1+1/2}$, $M_{2+1/2}$ omitted in $\sum M$,

$$a_R = \frac{3R^2\,(P' - 0)}{q\rho'\,(R^3 - R_4^3)} \rightarrow \frac{(P' - 0)}{q\rho'\Delta R} = \frac{P' - 0}{\rho'\,\frac{\Delta R}{2}} \quad \text{if } q = \frac{1}{2} \quad .$$

IGA-q. (87) gives $\vec{F}_{3+1/2}$, $\vec{F}_{4+1/2}$ in (65) with $P_1 = P_3 = 0$, so the conclusions of the general case with P = 0 and $M_{1+1/2}$, $M_{2+1/2}$ absent apply here. This means that $a_\alpha = 0$ if the α spacing is uniform and a_R is given by (66), replacing

$$\frac{1}{4}\sum \frac{\vec{F}}{M} \text{ by } \frac{1}{2}\sum \frac{\vec{F}}{M} \quad ,$$

$$a_R = \frac{3R^2\,(P' - 0)(1 + 1)}{2q\rho'\,(R^3 - R_4^3)} \rightarrow \frac{(P' - 0)}{q\rho'\Delta R} = \frac{P' - 0}{\rho'\,\frac{\Delta R}{2}} \quad \text{if } q = \frac{1}{2} \quad .$$

FGI-q. (88) gives $\vec{F}_1$, $\vec{F}_3$ in (67) with P = 0, so the general case holds. This means $a_\alpha = 0$ if the α spacing is equal, and (68) with $M_{1+1/2}$, $M_{2+1/2}$ terms absent, but using 1/4 $\sum M$ as in the general case, gives

$$a_R = \frac{3R^2 \left(P' - 0\right)\left(1 + 1\right)}{4q\rho' \left(R^3 - R_4^3\right)} \rightarrow \frac{\left(P' - 0\right)}{2q\rho'\Delta R} = \frac{P' - 0}{\rho' \frac{\Delta R}{2}} \quad \text{if } q = \frac{1}{4} \quad .$$

MAC-0 Mass Method. Use f = 1/2 and masses from (72). For all gradients, using arguments similar to those above, we get

$$a_R = \frac{P' - 0}{\rho' \frac{\Delta R}{2}} \quad .$$

XIV. CORNERS

In any mesh there are corner points (Fig. 23) which may fall into a number of categories or even mixed categories. For example, point 0 may be such that the side 0 - ℓ may be a free surface or constrained to move along a fixed line, and the same may be the case for the side 0 - (ℓ+1). Hence, there may be a mixture of conditions involved in deriving the gradients of point 0. It would be too lengthy to consider all the possible variations, so we will just indicate the general procedure, and consider a few special cases for Fig. 23.

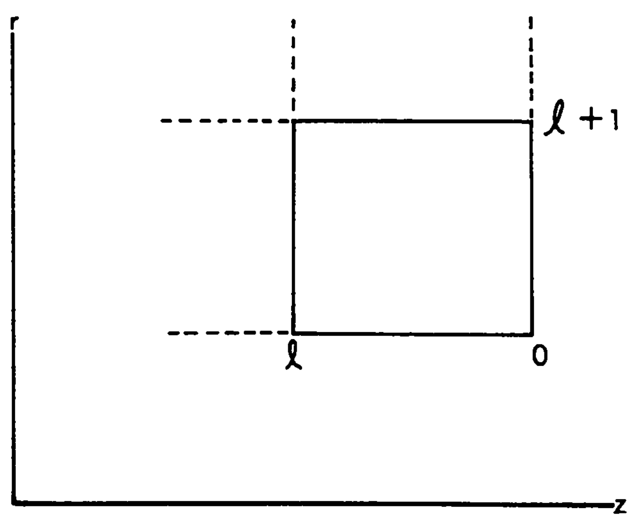

Fig. 23. A corner point.

We will use the general method, B3, applied in the two previous sections so that our assumptions will be

(1) For motion along a line ℓ (or ℓ+1), we assume that $P_\ell = P_{\ell+1/2}$ (or $P_{\ell+1} = P_{\ell+1/2}$) and set the accelerations normal to the line equal to zero.

(2) For motion along a line ℓ (or ℓ+1) to be a free surface, set $P_\ell = 0$ (or $P_{\ell+1} = 0$). It is interesting to note that IGT and IGA have the same general formulae for the case of a single zone. This remark is obvious, because for IGA we are averaging over only one zone, so

$$\sum \frac{\vec{F}}{M} = \frac{\vec{F}}{M} = \frac{\Sigma\vec{F}}{\Sigma M} \quad .$$

From (20b) applied to Fig. 23, the general formulae are:

IGT and IGA.

$$\vec{a} = \frac{\vec{F}_{\ell+1/2}}{M_{\ell+1/2}} = \frac{1}{M_{\ell+1/2}} \{\vec{i}\,\frac{\phi}{2}\,[(P_\ell - P_{\ell+1/2})(z - z_\ell)(r + r_\ell)$$

$$+ (P_{\ell+1} - P_{\ell+1/2})(z_{\ell+1} - z)(r_{\ell+1} + r)]$$

$$+ \vec{k}\,\frac{\phi}{2}\,[(P_\ell - P_{\ell+1/2})(r_\ell - r)(r_\ell + r)$$

$$+ (P_{\ell+1} - P_{\ell+1/2})(r - r_{\ell+1})(r + r_{\ell+1})]\} \quad . \tag{89}$$

FGI.

$$\vec{a} = \frac{1}{2}\left(\frac{\vec{F}_\ell}{M_{\ell+1/2}} + \frac{\vec{F}_{\ell+1}}{M_{\ell+1/2}}\right) = \frac{1}{2M_{\ell+1/2}}\,(\vec{F}_\ell + \vec{F}_{\ell+1})$$

$$= \frac{1}{2M_{\ell+1/2}} \{\vec{i}\,\frac{\phi}{2}\,[(P_\ell - P_{\ell+1/2})(z - z_\ell)(r + r_\ell)$$

$$+ (P_{\ell+1} - P_{\ell+1/2})(z_{\ell+1} - z)(r_{\ell+1} + r)]$$

$$+ \vec{k} \frac{\phi}{2} [(P_{\ell} - P_{\ell+1/2})(r_{\ell} - r)(r + r_{\ell})$$

$$+ (P_{\ell+1} - P_{\ell+1/2})(r - r_{\ell+1})(r + r_{\ell+1})]\} \quad . \tag{90}$$

Note that (90) is identical to (89) except for the factor 1/2. This factor is needed in FGI to counteract the use of $q = 1/4$ instead of $q = 1/2$ in the q-mass method and the use of (1/2)M in the MAC-0 mass method.

Use of these formulae for corner points in the plane, cylindrical, and spherical problems gives the same values for accelerations at a free surface as the general free surface point, provided the same values of f, q, M, etc., are used.

COMMENTS AND ACKNOWLEDGMENTS

All the work described in this report was done in 1961-1967. At that time the process of writing a report was started but never finished. Although there are copies of this unpublished work around, I have been asked at this time to publish a Los Alamos report. Although some work on this subject has been done by me and others since that time and some of the gradient calculations have been used in various codes, no attempt is being made to rewrite the report. Also, only a minor attempt is being made to revise the bibliography.

Much of the algebra and trigonometry of the derivations is quite lengthy, so in most places it has been omitted.

All the models discussed have been tried numerically and verify the analytical conclusions. All the programming for these tests was done by Karl B. Wallick. All the test problems were run by Karl B. Wallick and Leland R. Stein.

I would like to thank Eldon J. Linnebur and Dan E. Carroll for suggesting that I publish these notes as a report and for allowing me the time to do so. I would also like to thank George N. White, Patrick J. Blewett, Karl B. Wallick, and S. R. Orr for taking time to discuss two-dimensional Lagrangian hydrodynamics with me. Karen Knapp did the major part of the typing, which was very difficult, with assistance from Melissa Norris. Tessa Lippiatt guided me through the intricacies of getting a set of notes published as a report. Last but not least, I would like to thank Charlotte R. Hobart for promptly and efficiently retyping the final copy of this report in its entirety as it was necessary to input it into a different computer.

APPENDIX A

COMPARISON OF WAT TO INTRGRAL GRADIENTS

WAT, a code for calculating two-dimensional, Lagrangian, compressible hydrodynamics, devised by Walter Goad[9] is described in LAMS 2365. The forces on a vertex are derived in a very clever manner by calculating the resistance that the zones about the vertex offer to motion of the vertex, by D'Alembert's principle. The WAT code combines these forces (and arbitrarily defined masses) in a manner similar to IGT in (7). The objective here is to take the WAT gradients and see how they compare with gradients obtained from the integral methods.

A general point in the mesh is shown in Fig. 1 of LAMS-2365, and is reproduced below in our notation.

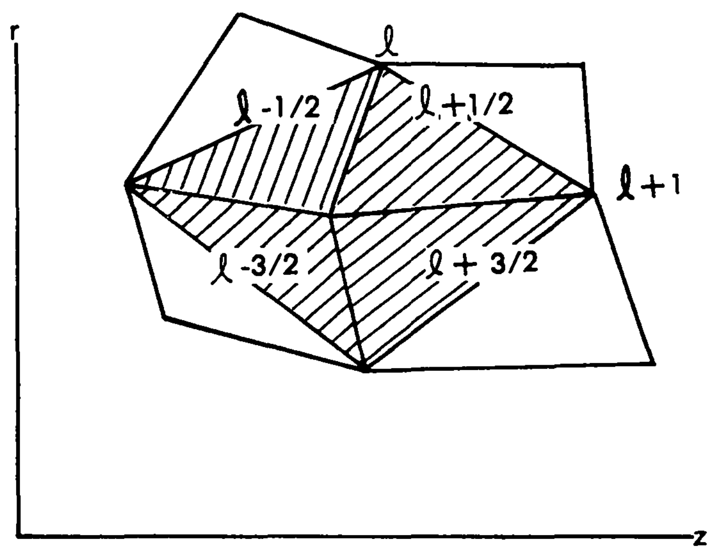

Fig. A-1. The WAT figure.

The acceleration is given as Eq. (2) of the report.

$$\vec{a} = \frac{\vec{F}}{M} = \frac{\sum_{\ell} \vec{F}_{\ell+1/2}}{\sum_{\ell} M_{\ell+1/2}} \quad , \qquad \text{(A-1)}$$

where the $M_{\ell+1/2}$ are taken as half the masses of the shaded portions of the figure, and the forces are given by formulae on pp. 20, 21 of LAMS-2365, which become in our notation

$$(F_r)_{WAT} = \frac{2\pi}{6} \sum_{\ell=1}^{4} \{P_{\ell+1/2}[(r + r_\ell + r_{\ell+1})(z_\ell - z_{\ell+1})$$

$$+ (r_\ell - r)(z_{\ell+1} - z) - (r_{\ell+1} - r)(z_\ell - z)]\} \tag{A-2}$$

and

$$(F_z)_{WAT} = \frac{2\pi}{6} \sum_{\ell=1}^{4} [P_{\ell+1/2}(r + r_\ell + r_{\ell+1})(r_{\ell+1} - r_\ell)] \quad . \tag{A-2a}$$

The angle of revolution about the z axis is 2π rather than the ϕ used in our work.

It is obvious from (A-1) that the WAT acceleration is of the type IGT, where the sum of the forces is divided by the sum of the masses. The general formulae for forces in IGT is given by (22) as (for revolution through angle ϕ),

$$(F_r)_{IGT} = \frac{\phi}{2} \sum_{\ell=1}^{4} (P_{\ell-1/2} - P_{\ell+1/2})(z - z_\ell')(r + r_\ell')$$

$$= \frac{\phi}{2} \sum_{\ell=1}^{4} \{P_{\ell+1/2} [(z - z_{\ell+1}')(r + r_{\ell+1}') - (z - z_\ell')(r + r_\ell')]\} \tag{A-3}$$

$$(F_z)_{IGT} = \frac{\phi}{2} \sum_{\ell=1}^{4} (P_{\ell-1/2} - P_{\ell+1/2})(r_\ell' - r)(r_\ell' + r)$$

$$= \frac{\phi}{2} \sum_{\ell=1}^{4} \{P_{\ell+1/2} [(r_{\ell+1}' - r)(r_{\ell+1}' + r) - (r_\ell' - r)(r_\ell' + r)]\} \quad . \tag{A-3a}$$

Recall that r_ℓ, z_ℓ in (A-2) are the coordinates of the point out at the end of side ℓ (while r'_ℓ, z'_ℓ in (A-3) represent points on side ℓ which are at some fraction, f, of the way. If we write (A-3a) in terms of

$$r'_\ell = r + f\,(r_\ell - r)$$

$$z'_\ell = z + f\,(z_\ell - z) \qquad \text{(A-4)}$$

we can perhaps then compare the two methods. If we look at F_z first (A-3a) and (A-4) give

$$(F_z)_{IGT} = \frac{\phi}{2} \sum_\ell \left[P_{\ell+1/2}\,({r'_{\ell+1}}^2 - {r'_\ell}^2)\right]$$

$$= \frac{\phi}{2} \sum_\ell \{P_{\ell+1/2}[r^2 + 2rf\,(r_{\ell+1} - r)$$

$$+ f^2\,(r_{\ell+1} - r)^2 - r^2 - 2rf\,(r_\ell - r) - f^2\,(r_\ell - r)^2]\}$$

$$= \frac{\phi}{2} \sum_\ell \{P_{\ell+1/2}[2rf(r_{\ell+1} - r_\ell) + f^2\,({r_{\ell+1}}^2 - 2rr_{\ell+1} + r^2 - r_\ell^2$$

$$+ 2rr_\ell - r^2)]\}$$

$$= \frac{\phi}{2} \sum_\ell \{P_{\ell+1/2}[f^2\,({r_{\ell+1}}^2 - {r_\ell}^2) + (2f - 2f^2)\,r\,(r_{\ell+1} - r_\ell)]\} \quad .$$

(A-5)

Now for this to give the same form as (A-2a), it is necessary that

$$f^2 = 2f - 2f^2 \text{ or } 3f^2 = 2f \quad .$$

Thus to achieve a WAT-like F_z from IGT one must use $f = 2/3$. Similarly (A-4) with (A-3) gives

$$(F_r)_{IGT} = \frac{\phi}{2} \sum_{\ell} P_{\ell+1/2}\{- f(z_{\ell+1} - z)[2r + f(r_{\ell+1} - r)]$$

$$+ f(z_{\ell} - z)[2r + f(r_{\ell} - r)]\}$$

$$= f^2 \frac{\phi}{2} \sum_{\ell} [P_{\ell+1/2}\{z(r_{\ell+1} - r_{\ell}) - z_{\ell+1} [r(\frac{2}{f} - 1) + r_{\ell+1}]$$

$$+ z_{\ell}[r(\frac{2}{f} - 1) + r_{\ell}]\} \quad . \tag{A-6}$$

Again, (A-2) can be written

$$(F_r)_{WAT} = \frac{2\pi}{6} \sum_{\ell} P_{\ell+1/2}\{z(r_{\ell+1} - r_{\ell}) - z_{\ell+1} (2r + r_{\ell+1})$$

$$+ z_{\ell}[2r + r_{\ell})]\} \quad . \tag{A-7}$$

These agree in form for $f = 2/3$.

As far as magnitudes of forces are concerned, from both (A-5) and (A-7) we can get for $f = 2/3$

$$F_{IGT} = f^2 \frac{\phi}{2} \frac{6}{2\pi} F_{WAT} = \frac{\phi}{2\pi} \frac{4}{3} F_{WAT} \quad . \tag{A-8}$$

General Conclusion.

If WAT were derived on a basis of angle revolutions ϕ rather than 2π, then the forces obtained by IGT with $f = 2/3$ would have the same algebraic form as those derived from WAT, and the magnitude would be larger by a factor of 4/3. In order to achieve exactly the same accelerations, one should then use for IGT 4/3 of the mass used by WAT, that is, $q' = 2/3$ of the masses of the shaded triangles in Fig. 24.

Note. I have no knowledge of the boundary cases used, so no analysis has been done on them. Also, we have done no numerical testing of this gradient. Further analytic work has shown that these gradients do not give spherical motion in an equally spaced spherical problem, as would be expected since the form is IGT.

APPENDIX B
FAMULARO-WHALEN GRADIENTS

In 1976, K. Famularo[10] and Paul Whalen[11] derived in very neat fashion, using Hamilton's principles, a set of gradients for 2-dimensional Lagrangian hydrodynamics. They came up with an IGT form of gradient. As might be expected with IGT, it was possible to prove analytically and numerically that this gradient does not give spherical motion in a spherical problem with equal angular spacing. I did not carry this work further.

APPENDIX C
SCHULZ GRADIENTS

I spent a good deal of time trying to write the Schulz[4] gradients in an integral form and almost made it, but not quite. However, the Schulz gradients have been used in many codes for a long time and do give spherical motion in a spherical problem. I was not able to prove this analytically.

APPENDIX D
ANOTHER PROPOSED GRADIENT (1986)

In this report it was suggested that the ICA-MAC-0 gradient might be a good one to try. Since that time (1964-1967), I have tried it in some quite complicated real problems. It worked fine in many problems, but had difficulty in one problem. This I traced back to the fact that one of the four subzones around a point had a much smaller mass than the other three subzones.

This meant one of the four $\vec{F}/M$ terms in Eq. (8) had a very small denominator, allowing that term to dominate the gradient calculation.

In the process of writing this report, it has occurred to me that one way to prevent the above difficulty would be to use the FGI technique of taking zones in pairs, but in an IGA-MAC-0 type model. This I am calling IGAP-MAC-0 (Integral Gradient Average of Pairs) just for the sake of a name.

This would give an expression of the type of Eq. (9), that is

$$\vec{a} = \frac{1}{L} \sum_{\ell=1}^{L} \vec{a}_{\ell} = \frac{1}{L} \sum_{\ell=1}^{L} \frac{\vec{F}_{\ell-1/2} + \vec{F}_{\ell+1/2}}{M_{\ell-1/2} + M_{\ell+1/2}}$$

but with the masses as the MAC-0 masses and the forces given by applying (20b) twice, once to each mass of the pair and adding these expressions together before dividing by the sum of the masses.

I am quite positive that this technique would eliminate the difficulty and give good motion in plane, cylindrical, and equi-angular spaced spherical problems.

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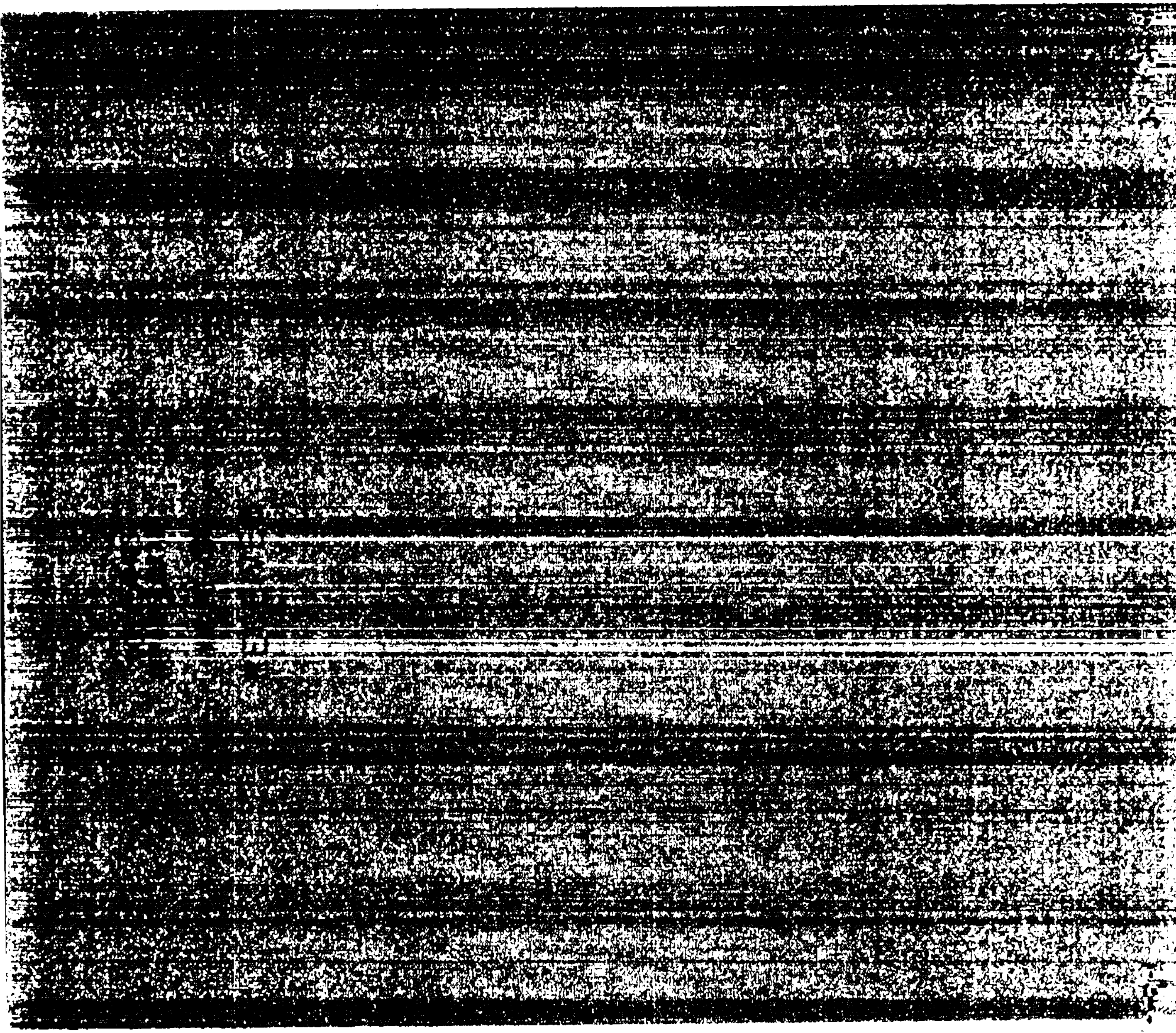

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